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Characterizing metacognitive processes and mathematical skills during problem solving by using a non-linear orientation base as a self-regulation tool

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Abstract
Over the past few decades, research has shown that metacognition plays a critical role in helping students in the solving of mathematical problems. Despite the existence of several tools for assessing the metacognitive processes in problem solving, there is a notable lack of tools designed to enhance these processes over time. The purpose of this study is to introduce an instrument called Non-Linear Orientation Base (N-LOB) and examine its impact on metacognitive self-regulation during pattern-based problem solving. In particular, the study aims to characterise the mathematical skills and metacognitive processes shown by three groups of sixth grade students as they construct and refine a N-LOB. Results suggest that metacognitive regulation processes have a general character when compared to the awareness and evaluation processes, which are more closely tied to specific skills required for pattern-based problem solving. Moreover, deeper engagement with the N-LOB correlates with the emergence of more advanced mathematical skills related to generalization. The study concludes by highlighting the potential of the N-LOB as a tool for promoting metacognitive self-regulation in the context of problem solving.

Keywords Problem solving · Mathematical patterns · Metacognition · Self-regulation · Non-linear orientation base

1 Introduction

One of the central processes of mathematical competence is problem solving. This process, which includes several sub-processes such as understanding the statement, planning and executing a strategy, or verifying the solution, remains one of the most present topics in mathematics education research (Schoenfeld, 2014). This presence has been reinforced by

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the findings of the most recent PISA report (OECD, 2023), which points to widespread underperformance in problem solving. At the same time, one of the aspects that makes the task of problem solving more complex is metacognitive self-regulation, that is, the awareness and management of mental processes that allow one to regulate, control, and evaluate decisions when solving a problem (Özsoy & Ataman, 2017).

While the literature provides tools for understanding students' regulation, control, and evaluation processes during problem solving (Çini et al., 2023; He et al., 2024; Wider & Wider, 2023), there are limited tools available to help students improve these processes over time. Moreover, the existing metacognitive tools designed for this purpose—such as planning sheets (An & Cao, 2014), the reflection assistant (Gama, 2001), or the self-monitoring questionnaire proposed by Goos et al. (2000)—are often tailored for research use, which may reduce their relevance for students. To address this gap, this study introduces a metacognitive self-regulation tool called the Non-Linear Orientation Base (N-LOB), which aims to facilitate the structuring and articulation of metacognitive processes during problem solving (Torregrosa et al., 2020). This tool, in contrast to those mentioned above, is constructed by students based on their own analysis of the metacognitive processes they develop while solving a problem. In addition, the Non-Linear Orientation Base is not a tool that is created and remains unchanged, but rather evolves as students solve new problems and thus develop more complex metacognitive processes.

In this context, the aim of the study is to analyse the influence of creating and modifying this tool on the development of metacognitive self-regulation processes during pattern-based problem solving.

2 Theoretical framework

The theoretical framework is divided into two distinct sections. The first section outlines the theoretical foundations related to problem solving and mathematical self-regulation. The second section presents the core features of the Non-Linear Orientation Base (N-LOB) and justifies its use. Finally, the study objectives are stated in relation to the theoretical foundations discussed.

2.1 Problem solving and mathematical self-regulation

Current curricula around the world emphasise competency-based learning as the true essence of engaging with mathematics (LOMLOE, 2020; NCTM, 2000; Common Core, 2010). The PISA assessment framework (OECD, 2017) defines mathematical competence as "an individual's ability to formulate, use and interpret mathematics" (p. 66). This competence is structured around different processes that students need to develop (Niss & Højgaard, 2019), one of which is problem solving. Problem solving is understood as any situation for which the solver does not have an immediate strategy for finding a solution (Schoenfeld, 2014). As noted by Mayer (2002) and Foster (2023) inspired by Pólya (1945), problem solving can also be referred to as the solution of non-routine tasks that require the solver to: understand the problem, developing a plan, carrying out the plan and looking back verifying them (Pólya, 1945). In contrast, routine tasks, also known as exercises, encourage imitation of previously taught procedures or methods (Liljedahl, 2016). In this article, we will refer to problems as non-routine activities or tasks.

During the problem solving process—and thus during the development of connections (Maaß, 2006), reasonings (Stylianides, 2008) and representations (Mainali, 2021)—the solver engages in a number of metacognitive mechanisms, following highly complex mental processes (Schoenfeld, 1992). Metacognition can be defined as the awareness and control of one's own learning (Zimmerman, 1990), meaning that we are aware of what we think and how we think. Wilson and Clarke (2004) identified three metacognitive processes that occur during problem solving: metacognitive regulation, metacognitive awareness and metacognitive evaluation.

- *Metacognitive regulation* occurs when individuals use their metacognitive skills to guide their thinking and knowledge. This regulation is based on individuals' knowledge of themselves and their strategies, including how and why they use particular strategies. It also involves executive skills such as planning, self-correction and goal setting to optimise their cognitive resources.
- *Metacognitive awareness* refers to an individual's understanding of their progress in the learning process or problem solving, as well as their content knowledge and awareness of their personal learning or problem solving strategies. It involves recognising what needs to be done, what has been done, and what could be done in specific learning contexts or problem solving scenarios.
- *Metacognitive evaluation* involves making judgments about one's own cognitive processes, capacities and limitations in a given situation or as personal attributes. For example, individuals may evaluate the effectiveness of their thinking or the appropriateness of their strategy choices.

These metacognitive processes have been refined over time and applied specifically to non-routine problem solving. For example, Ader (2019) and Whitebread et al. (2009) suggest that the regulatory process can be understood as planning, while awareness is associated with monitoring and controlling. Thus, these authors consider four metacognitive processes during problem solving: planning, monitoring, controlling and evaluating.

In recent years, building on the work of Wilson and Clarke (2004), researchers such as English (2016) and Geiger et al. (2021) have developed the concept anticipatory metacognition. This refers to the ability to recognise possible ways to approach an unstructured problem by relating it to other contexts and situations. In this sense, anticipatory metacognition can be understood as the use of real-world references for solving non-routine problems. This anticipatory process can occur before and during the problem-solving process.

Several studies have shown that students who can effectively self-regulate their metacognitive processes achieve higher performance in mathematics (Coutinho, 2007) and show greater proficiency in mathematical competence, especially when developed collaboratively (Schnaubert & Bodemer, 2019). In this regard, metacognitive self-regulation can be fostered over time through activities and tools that promote regulation, awareness and evaluation processes (Papaleontiou-Louca, 2008).

Several studies have focused on the development and analysis of tools to enhance metacognitive self-regulation. First, there are *planning sheets* (An & Cao, 2014), which aim to provide a checklist of metacognitive processes that students should verify during problem posing and problem solving tasks. These processes, outlined by the researchers, allow students to verify aspects such as whether the design is appropriate, efficient, and considers the application of complex strategies. As noted by the authors at the end of their study, both students who use the planning sheets and those who do not, show similar strategies in terms of awareness and evaluation. The researchers conclude that this similarity may be

due to two factors: the planning sheets were created by the researchers and therefore lacked meaning for the students (Ge & Land, 2003), or the implementation period was too short.

Secondly, there is Gama's (2001) *Reflection Assistant*, a software designed to scaffold students' metacognitive processes during problem solving. This software is based on an analysis of the processes that students engage in when solving a problem. To achieve this, researchers use reflective questions that help train the assistant and automate support in the long term. Although the Reflection Assistant shows potential, particularly in facilitating self-regulation of cognitive processes, the researchers point out that for some students, the reflective questions themselves seem to be more effective than the software.

Thirdly, there is the *self-monitoring questionnaire* proposed by Goos et al. (2000), which examines students' ability to identify metacognitive "red flags", or moments when they might get stuck in problem solving. This questionnaire, which includes items related to monitoring and awareness processes, is completed by students after solving a non-routine problem—in this case, "The Money Problem". As the authors point out, the results of using this questionnaire are directly related to students' problem solving strategies, i.e. students who show better self-regulation tend to use more appropriate strategies. Conversely, the authors also emphasise that the use of this tool alone does not guarantee true awareness of self-regulatory processes, nor does it guarantee long-term improvement.

The aforementioned studies, along with other research (Çini et al., 2023; Clarke, 1989; Loh & Lee, 2019; Özsoy & Ataman, 2017; Wilson & Clarke, 2004), unanimously highlight the difficulty of identifying one or more generalisable metacognitive cycles in a given problem that apply uniformly to all problem solvers and characterise processes that always occur in the same way and at the same time (Goos et al., 2000). This observation leads to two important assumptions. First, analysing the metacognitive processes that students engage in during problem solving can be key to understanding success and/or failure in solving a problem (Çini et al., 2023). Second, analysing students' metacognitive processes requires the use of multiple data collection instruments as well as careful triangulation, since metacognition is a mental and subjective process that is often difficult to verbalise clearly (Godino & Llinares, 2000).

2.2 The non-linear orientation base

The Non-Linear Orientation Base (N-LOB) is a metacognitive self-regulation tool based on the concept of the Orientation Base of Action, first developed by Nunziati (1990) and further developed by Sanmartí and Jorba (1993). These authors conceptualised the Orientation Base of Action as a tool to promote students' anticipation and planning of the steps necessary to solve a routine situation or problem in the scientific-mathematical domain. Examples include solving a first degree equation or classifying triangles according to their sides and angles. Their studies showed that the Orientation Base of Action helped students to self-regulate more effectively in these types of tasks and to self-evaluate their skills better.

Later, Villalonga and Deulofeu (2017) and Villalonga and Andrews (2018) created and applied an Orientation Base of Action during non-routine problems, following the guidelines provided by Sanmartí and Jorba (1993). Their main objective was to characterise students' mathematical competence through the process of creating and applying an Orientation Base of Action. At the end of their study, the authors noted that while the potential of the tool was evident, its step-by-step, linear format made it difficult for students to use

because the problem solving process is often not sequential or linear, although the written solution at the end is.

Building on the contributions of Sanmartí and Jorba (1993) and addressing the limitations identified by Villalonga and Deulofeu (2017), the authors of this study conceptualised the Non-Linear Orientation Base (Torregrosa et al., 2021): a mathematical self-regulation tool designed and implemented by students during problem solving tasks. The N-LOB serves as an orientation guide (as illustrated in Fig. 1) that integrates both questions related to *metacognitive processes* (Clarke, 1989) and those related to the development of *mathematical skills*.

By *mathematical skills* we mean cognitive products—such as concepts, algorithms or mathematical tools—that students acquire during their compulsory education, including algorithms for basic calculations, multiplication tables, statistical definitions or the production of double-entry tables (Puig, 1996).

This tool has a graph structure with four main nodes corresponding to the four phases of problem solving (Pólya, 1945). The N-LOB draws parallels between these problem solving phases and the planning, monitoring, awareness and evaluation processes identified by Ader (2019). Specifically, we understand that Ader's "planning" process corresponds to Pólya's "understanding and developing a plan" phases. Ader's "monitoring and awareness" processes are associated with Pólya's "carrying out the plan" phase, and finally, Ader's "evaluation" process is linked to Pólya's "looking back" phase. Within each node, students are asked to add statements or questions related to the mathematical skills and metacognitive processes they use when solving a problem.

The aim of this tool is to help students become aware of the metacognitive processes they are developing and to articulate them explicitly, thereby facilitating self-regulation

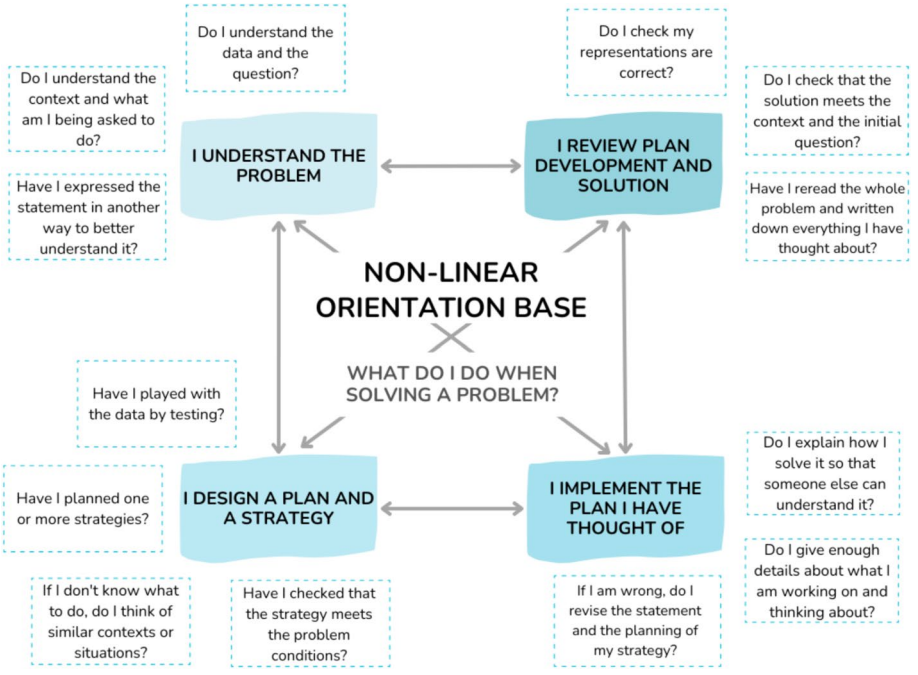


Fig. 1 Example of a generic non-linear orientation base

during problem solving. This not only helps to solve the problem efficiently, but also plays a crucial role in the process of communicating and representing strategies, connections and reasoning (Goos et al., 2000).

The process of creating the non-linear orientation base (N-LOB) is structured into several cycles (see Fig. 2) and begins with *Cycle 1*, in which students are individually presented with an initial problem of low difficulty. This *initial problem* is designed to involve mathematical skills that the problem solver has previously worked on. After solving the problem individually, a *group sharing* is conducted during which various students present their solution strategies to the entire class. The teacher's role in this context is to facilitate the discussion by validating the proposed solutions and paraphrasing students' contributions when their ideas are not clearly articulated. Next, students carry out a *self-assessment* using a metacognitive self-regulation questionnaire (Goos et al., 2000). The aim of this questionnaire is to get students to explicitly describe the processes and skills they used at each step, which they may not have verbalised during their written problem-solving process (Villalonga & Deulofeu, 2017). This is followed by a *peer-assessment*. For this purpose, the self-regulation questionnaire is adapted to the third person singular so that a student can apply the same self-regulation questions to a peer's problem solving. The purpose of peer assessment is to help students identify the processes and skills used by others.

At the end of Cycle 1, a *team discussion* is held with all the students who have faced the same problem to create the N-LOB.1. At the beginning of this *team discussion*, the teacher draws on the board a diagram similar to Fig. 1, noting only Pólya's (1945) problem solving phases and leaving the remaining elements of the framework blank for students to complete. To fill in these elements, the teacher asks all students which metacognitive processes and mathematical skills they identified in the self-assessment and peer-assessment questionnaires. As a group, they select the processes and skills that are commonly shared among the majority of students or those mentioned by an individual student but positively valued by their peers. These processes and skills are then organized on the board according to Pólya's (1945) problem solving phases. This discussion leads to the creation of the first

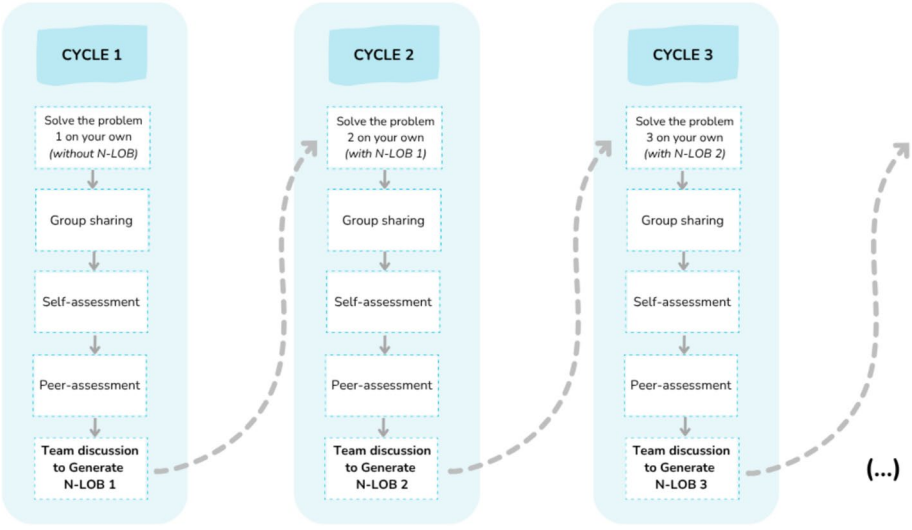


Fig. 2 Creation and development cycles of a non-linear orientation base (N-LOB)

Non-Linear Orientation Base (*N-LOB.1*), as illustrated earlier. Once this first base has been developed, students are encouraged to document it individually, either physically or digitally, in order to facilitate a deeper internalisation (Sanmartí, 2010).

With the *N-LOB.1* in hand, students move on to *Cycle 2*, where they use this first version of the base to *solve a second problem* of slightly higher difficulty. During this cycle, students can check their *N-LOB.1* as many times as they need to. The purpose of this multiple checking is to assess which metacognitive processes and mathematical skills might be useful in solving the second problem, thus enabling more effective self-regulation. After solving the second problem, a *group sharing* occurs. Later, students carry out another *self-assessment and peer assessment* to make explicit the skills and metacognitive processes they used. Finally, another *team discussion* is held in which students analyse the *N-LOB.1* they used for the second problem and identify the processes and skills they want to modify, include or remove in order to create the second version of the Non-Linear Orientation Base (*N-LOB.2*).

The number of cycles of problem solving, group sharing, self-assessment, peer assessment and team discussion used to create and modify the N-LOB depends on the desired depth and specificity of the base. On the basis of the information given above, it seems reasonable to assume that the more cycles carried out, the more specific and detailed the base becomes but so far, we have no previous studies to support this statement yet. Ideally, each student would develop his or her own N-LOB; however, if the participating group has not previously worked with the tool, it is recommended to develop a single group base that serves as a model for the gradual refinement of the tool (García & Sanmartí, 1998).

3 Goals of the study

The theoretical framework presented, and the background of this study lead us to claim that the creation and modification of a N-LOB fosters the problem solver's metacognitive self-regulation during problem solving. To confirm the validity of this statement, we defined the following research objective: To analyse and characterise the mathematical skills and metacognitive processes shown by three different groups of 6th grade elementary school students during the development and use of a Non-Linear Orientation Base in order to explore its potential for promoting metacognitive self-regulation in problem solving.

4 Methodology

4.1 Participants

The study data were collected from two schools in the Barcelona metropolitan area, involving three groups of sixth-grade students, referred to as groups A, B and C. Group A, with 24 students, belongs to a private school that has been working on routine problem solving for one hour per week since the 2017–2018 school year. Groups B and C, with 26 students each, come from a public school that focuses on routine and non-routine problem solving during one or two sessions per month, using problems from the Generalitat de Catalunya competency tests (Consell superior d'avaluació, 2013). The selection of these two schools is due to the researchers' intention to include a sample of students with limited problem solving experience, thereby minimizing potential interference from prior exposure

to practices that foster long-term metacognitive awareness. Although the two schools do not dedicate the same amount of time to problem solving (since groups B and C perform non-routine problem solving in one or two sessions per month, in contrast to group A), we consider that all three groups have limited experience with non-routine problem solving. In this sense, while the three groups are not entirely analogous, we believe that their limited experience with problem solving makes them ideal participants for investigating whether the creation and modification of a N-LOB fosters the problem solver's metacognitive self-regulation during problem solving.

4.2 Instruments

To create a N-LOB with groups A, B and C, we selected four pattern problems (see Fig. 3). We chose this mathematical content for two reasons. First, neither school had previously worked with this type of problem on a regular basis. Second, the four problems meet the criteria to be considered mathematically rich. As noted by Gojak (2017), mathematically rich activities include a high level of cognitive demand and encourage students to make connections between concrete and abstract mathematical concepts, which in turn promotes the emergence of complex metacognitive processes.

To sequence the four problems from least to most complex, we considered both the difficulty of finding the general term and the mathematical skills involved in the problems. Therefore, unlike the first three problems, which are arithmetic progressions of increasing complexity, the last problem is a geometric progression. This choice allows for the

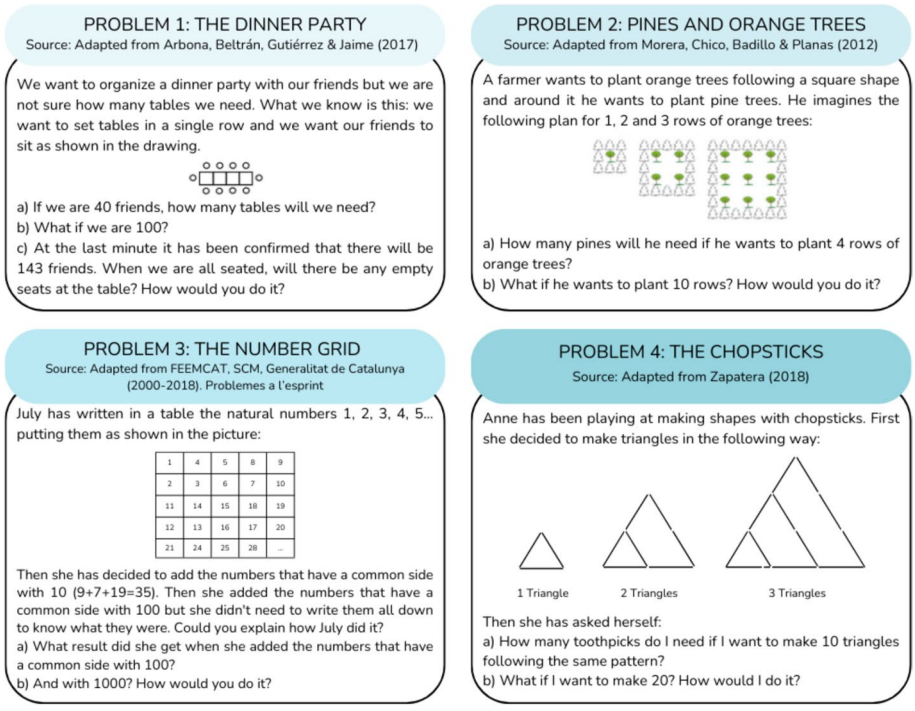


Fig. 3 Pattern problems used in this study

Table 1 Self-assessment and peer-assessment questionnaire items in relation to the processes identified by Ader (2019)

Items of the self-assessment questionnaire	Items of the peer-assessment questionnaire	Metacognitive referral process (Ader, 2019)
1. After you have finished reading the statement for the first time, what is the first thing you thought you could do to solve it? Just tell us the first idea that came to your mind.	1. After reading and analysing your partner's resolution, what do you think was the first idea he or she had to solve the problem?	Planning
2. What mathematical knowledge do you think you have used to solve the problem?	2. What mathematical knowledge do you think your partner has used to solve the problem?	Monitoring
3. While solving the problem, was there a moment in which you changed the way you were solving it? At what point and why did you change what you had been thinking?	3. Do you think there has been a moment in which your partner has changed the way he/she was solving the problem? At what point do you think it was and why do you think he/she changed the strategy?	Control and evaluation
4. Do you think you have checked the problem while you were doing it, when you have finished, or at all times? What have you checked (the calculations, the drawings, the statement, what you have written…)?	4. Do you think your partner has checked the problem while he/she was doing it, when he/she has finished or at all times? What do you think he/she has checked (the calculations, the drawings, the statement, what he/she has written…)?	Evaluation

emergence of more complex mathematical skills and metacognitive processes (Villalonga & Deulofeu, 2017). In addition, as shown in Fig. 3, the written explanations of the problems include pictorial representations to facilitate comprehension (Samson & Schäfer, 2007).

To design the self- and peer-assessment questionnaires, we developed four questions (Table 1) that focused on the metacognitive processes identified by Ader (2019) and therefore, aligned with Pólya's (1945) problem solving phases. Each question aimed to promote the emergence of specific sub-processes and mathematical skills as outlined by Goos et al. (2000). Both questionnaires were designed to encourage students to reflect on their own self-regulation processes, which is crucial for modifying the N-LOB in each cycle.

4.3 Instructional intervention and data collection

The procedure was designed as an intervention and was implemented by the first author of this study without the participation of the classroom teachers of groups A, B and C. This decision was made because there were no previous examples of the use of a N-LOB and the researchers wanted to ensure the objectivity of the data collected, especially during the discussions that facilitated the creation and modification of the N-LOB.

A total of four cycles of problem solving, group sharing, self-assessment, peer assessment and team discussion were conducted, resulting in three modifications of the N-LOB (see Appendix). Each cycle consisted of a two-hour session divided as follows:

- 30 min for reading and solving the problem individually.
- 20 min for sharing strategies and results together.
- 40 min for self-assessment and peer-assessment.
- 30 min for group discussion and creation or modification of the N-LOB.

The intervention was conducted with independent groups of students, each participating in four weekly two-hour sessions over the course of a single month.

During the problem solving sessions and the self- and peer-assessment activities, the researcher only made neutral comments when students encountered difficulties. These comments, such as "reread the question", "keep thinking about what you could do" or "think about what you have done", were intended to avoid influencing the natural development of the students' metacognitive processes (Jacobse & Harskamp, 2012). During the group sharing (to discuss problem solving strategies), as well as in the group discussion (to create or modify the N-LOBs), the researcher employed guiding questions such as: "What did you do to solve the first/second/third question?", "What did you notice in question…?", and "At what stage would you like to add this process/skill?". The purpose of these questions was to facilitate both discussions without introducing any information that had not been explicitly mentioned by the students. It was essential that students were consistently made aware of the correct solution to the problem, as well as the range of problem solving strategies proposed by their peers to arrive at that solution.

After carrying out the above procedure with groups A, B and C, we obtained three different types of data. First, we had the written solutions to the four problems that the students had solved, so we had 76 solutions (24 from group A, 26 from group B, and 26 from group C) for each problem. Second, we collected 76 self-assessment questionnaires and 76 peer-assessment questionnaires for each problem, 304 questionnaires for each assessment (608 questionnaires in total). Finally, we had three versions of the N-LOB created by each of the groups A, B and C.

5 Data analysis

In this section we present the analysis of the N-LOBs generated to achieve the research goal. To achieve this goal, we have organised the three N-LOB generated by groups A, B and C in a comparative tabular format to accurately visualise their evolution across cycles. It is important to note that the N-LOB has a graph structure, which makes it difficult to compare changes between versions at a glance. Therefore, we created a double-entry table where each column corresponds to a development cycle (N-LOB 1, 2 and 3) and each row corresponds to a specific skill or process. We then categorised the skills and metacognitive processes identified in the N-LOBs.

To categorise the skills, we have based our approach on the skills identified by Schoenfeld (2007), namely: facts, procedures, definitions and concepts. Given the nature of the selected problems, we consider that facts, definitions and concepts can be grouped into a single category. In contrast, procedures should be distinguished between algorithms and mathematical tools. The skill categories are defined as follows:

- *Knowledge of facts and definitions* (FD): Skills involving theoretical knowledge or mathematical definitions, such as understanding the definition of a sequence or a pattern.
- *Knowledge of algorithms* (A): Skills involving the application of mathematical algorithms. For example, performing addition or division.
- *Knowledge of mathematical tools* (MT): Skills involving understanding and using mathematical tools, such as creating a graph, drawing or creating a table.

In terms of metacognitive processes, we used Clarke's (1989) classification, which distinguishes between:

- *Regulatory Processes* (RP): processes that enable students to be aware of the decisions necessary to plan and implement problem solving processes, such as selecting relevant problem data.
- *Awareness Processes* (AP): processes that enable students to monitor the effectiveness of their actions or decisions in terms of strategy, such as drawing a pattern or trying to generalise.
- *Evaluation Processes* (EP): Processes that involve evaluating the situation and evaluating the solution, such as checking that the explanation is clear.

To characterise metacognitive processes and mathematical skills, the two researchers in the study analysed each item in each group's N-LOB and linked it to both a metacognitive process and a mathematical skill where appropriate. Subsequently, they compared their analyses and found a 92% match. Finally, the 8% of cases where there was disagreement were resolved through discussion in order to reach a verdict and obtain the definitive characterisation of each item in the N-LOBs. For example, if a student noted an item such as "highlight the data and the question", this was understood as a regulatory process (RP) because this decision helped the student to be aware of how they were planning the problem solving process. If an item such as "draw the continuation of the pattern" was noted, it was characterised as an awareness process (AP) because it involved using a strategy that allowed students to understand the pattern (FD) and use a mathematical tool (MT) such as drawing. Finally, if a student noted an item such as "check the calculations to make sure they match the problem statement", it was characterised as an evaluation process (EP), because it involved checking the problem solving strategy against the initial data using algorithms (A).

6 Results

In order to facilitate the understanding of the results, we coded each metacognitive process or skill that appeared in the N-LOBs of the different groups using a three-number system. The first number indicates the cycle in which the N-LOB was developed (1, 2 or 3). The second number corresponds to Pólya's (1945) problem solving phases: 1. Understanding, 2. Planning, 3. Executing, 4. Reviewing. The third number indicates the ascending order of the process or skill within the problem solving phase: 1. First item, 2. Second item, 3. Third item, and so on. For example, an item coded 3.2.4 belongs to the third version of the N-LOB (3), planning phase (2), fourth item (4). Under each item, we identified the type of

metacognitive process the students were engaging in according to Clarke's (1989) classification. In addition, if the process was related to a specific skill, it was noted according to the definition detailed above.

Table 2 shows the three Non-Linear Orientation Bases produced by the students in group A.

With regard to the N-LOB developed by the students in group A, we first note that the group decided not to separate Pólya's (1945) phases 2 and 3: planning and execution although these phases were separated during the group discussion on the board, the students decided to place the same items in phases 2 and 3 and ultimately merged these two phases. In this sense, the students felt that both phases involved interrelated metacognitive processes. Therefore, it made sense to keep them together. As noted by Goos et al. (2000), this occurs when students perceive that the core of problem solving lies in planning and executing, as they continually adjust the plan as they implement it.

Second, we observe that there are not only awareness processes but also evaluation processes within the "planning and execution" phases. This observation suggests that problem solving is indeed not a linear process and, contrary to what might be expected, metacognitive processes related to evaluation do not appear only during the reviewing phase (Villalonga & Andrews, 2018). Furthermore, we also see that there are regulatory processes during the reviewing phase, which are included in item 1.4.5 (try to think of other ways to solve the problem). This item, which is consistent across the three cycles of the N-LOB, shows how students in group A propose to check the statement at various times in order to outline new strategies. As shown in Table 3, student GA21 changes her approach to solving the first problem, transitioning from drawing (to determine the number of tables required for 40 people) to calculating (to ascertain the number of tables needed for 100 people). In her self-assessment questionnaire, she states that she made this change in the strategy because the problem can be solved in different ways and calculating is faster than drawing. This sub-process of problem solving is not typically seen in younger problem solvers. It is more likely to emerge naturally in expert problem solvers (Schoenfeld, 1992).

When analysing the first phase, "understanding" the problem, we find that it is very general in nature, as all the regulatory processes identified are not linked to specific mathematical skills related to pattern problems. This suggests that for group A, understanding the problem does not depend significantly on the content being dealt with.

When analysing the "planning and executing" phases, we see that in *N-LOB.1A*, the references are generally to awareness processes related to the use of general mathematical tools, such as making a drawing. In *N-LOB.2A*, process 2.3.3 (think about the pattern) appears, where students refer explicitly to identifying the pattern, which is a type of factual knowledge specific to pattern problems. In *N-LOB.3A*, item 3.3.1 appears as a combination of items 2.3.1 and 2.3.3. In this item 3.3.1, students explicitly refer not only to identifying the pattern, but also *continue the pattern and draw it*. The development of these items shows that the "planning and executing" phases increasingly focus on developing metacognitive processes and mathematical skills specific to pattern problems. Thus, they move from a more general state to a more content-related one.

When analysing the "reviewing" phase, we find that it has changed very little between *N-LOB.1A* and *N-LOB.3A*. The two items that show the most development are 1.4.1 and 1.4.4. Item 1.4.1 is modified in *N-LOB.2A* (item 2.4.1), indicating the checking not only of calculations, but also of the *continuation of the pattern* (Facts and Definitions + Mathematical Tools). Item 1.4.4 is modified in both *N-LOB.2A* (2.4.4) and *N-LOB.3A* (3.4.4). In 2.4.4, students note that the solution should be *clean and well written*. In 3.4.4, students add *and can be understood by another person*. As shown in Table 4, the modification of

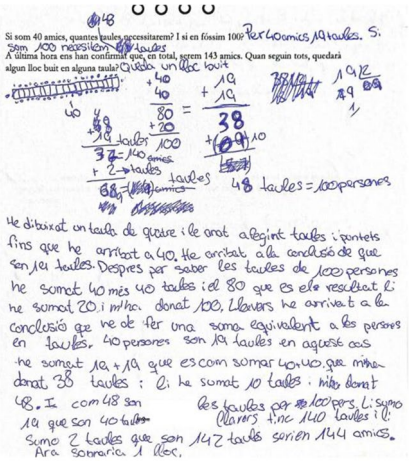
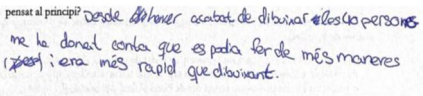
Table 2 Non-linear orientation bases—group A

	N-LOB.1A	N-LOB.2A	N-LOB.3A
Understanding	1.1.1 Read multiple times to understand the problem *(Regulatory Process)*	2.1.1 Read multiple times to understand the problem *(Regulatory Process)*	3.1.1 Read multiple times to understand the problem *(Regulatory Process)*
	1.1.2 Understand what the drawing means *(Regulatory Process)*	2.1.2 Understand what the drawing means *(Regulatory Process)*	3.1.2 Understand what the drawing means and the pattern it follows *(Regulatory Process)*
	1.1.3 Read between the lines *(Regulatory Process)*	2.1.3 Read between the lines to see what the problem suggests *(Regulatory Process)*	3.1.3. Read between the lines to see what the problem suggests *(Regulatory Process)*
	1.1.4 Take the data and the question *(Regulatory Process)*	2.1.4 Take the data and the question *(Regulatory Process)*	3.1.4 Take the data and the question *(Regulatory Process)*
		2.1.5 Break the problem into parts *(Regulatory Process)*	3.1.5 Break the problem into parts *(Regulatory Process)*
Planning and executing	1.2.1 Think about the calculation or the drawing I have to do *(Awareness Process of Algorithms and Mathematical Tools)*	2.2.1 Think about the calculation or the drawing I have to do *(Awareness Process of Algorithms and Mathematical Tools)*	3.2.1 Think about the calculation or the drawing I have to do *(Awareness Process of Algorithms and Mathematical Tools)*
	1.3.1 Draw what I am being asked to draw *(Awareness Process of Mathematical Tools)*	2.3.1 Draw what I am being asked to draw *(Awareness Process of Mathematical Tools)*	3.3.1 Continue the pattern and draw it *(Awareness Process of Facts, Definitions and Mathematical tools)*
	1.3.2 Review statement and keywords *(Evaluation Process)*	2.3.2 Review statement and keywords *(Evaluation Process)*	3.3.2 Review statement and keywords *(Evaluation Process)*
		2.3.3 Think about the pattern *(Awareness Process of Facts and Definitions)*	
Reviewing	1.4.1 Draw to check that I have done the calculations correctly *(Evaluation Process of Mathematical Tools)*	2.4.1 Draw to check that I have done the calculations or the continuation of the pattern correctly *(Evaluation Processes of Algorithms, Mathematical Tools, Facts and definitions)*	3.4.1 Draw to check that I have done the calculations or the continuation of the pattern correctly *(Evaluation Processes of Algorithms, Mathematical Tools, Facts and definitions)*

Table 2 (continued)

	N-LOB.1A	N-LOB.2A	N-LOB.3A
	1.4.2 Check the calculations *(Evaluation Process of Algorithms)*	2.4.2 Check the calculations *(Evaluation Process of Algorithms)*	3.4.2 Check the calculations *(Evaluation Process of Algorithms)*
	1.4.3 Redoing the calculations *(Evaluation Process of Algorithms)*	2.4.3 Redoing the calculations *(Evaluation Processes of Algorithms)*	3.4.3 Redoing the calculations *(Evaluation Processes of Algorithms)*
	1.4.4 Check that the solution makes sense *(Evaluation Process)*	2.4.4 Check that the solution makes sense, is clean and well written *(Evaluation Process)*	3.4.4 Check that the solution makes sense, is clean, well written and can be understood by another person *(Evaluation Process)*
	1.4.5 Try to think of other ways to solve the problem *(Regulatory Process)*	2.4.5 Try to think of other ways to solve the problem *(Regulatory Process)*	3.4.5 Try to think of other ways to solve the problem *(Regulatory Process)*

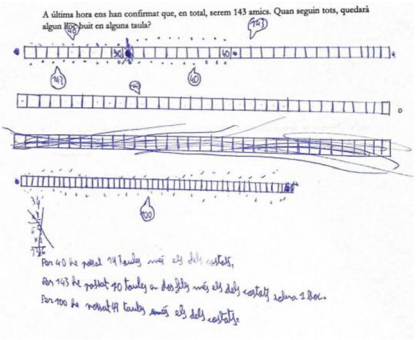
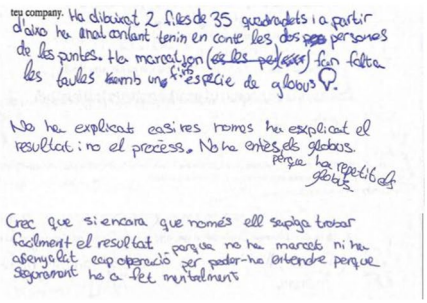
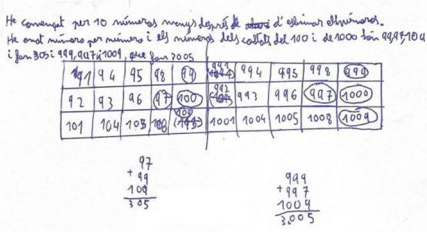
Table 3 GA21's solution to problem 1 and response to question 3 of the self-assessment questionnaire

Original production	Description and translation
	Translation of the student's text: *I drew a table for four and kept adding tables and dots* [referring to people] *until I reached 40. I concluded that it takes 19 tables. Then, to find out the number of tables for 100 people, I added 40 plus 40 tables, and to the 80 that is the result, I added 20 to make 100. So, I concluded that I have to make an addition equivalent to the people but in tables. 40 people are 19 tables, so in this case, I added 19 + 19, which is like adding 40 + 40, and I got 38 tables, then I added 10 tables and got 48. Since there are 48 tables for 100 people, I add 19 which are 40 tables, so I have 140 tables, and then I add 2 tables, making 142 tables and 144 friends* [she writes in the statement that there is one free seat]. Problem description GA21: The student draws 19 tables for 40 people in the upper left corner of the page. She then calculates 40 + 40 = 80 and 80 + 20 = 100 to reason how many tables are needed for 100 people. Knowing that 19 tables are needed for 40 people, she adds 19 + 19 = 38 for 80 people. Finally, she divides 19 by 2, gets 9 remainder 1, adds one more table, and sums 38 + 10 = 48. However, she omits 1 table needed to seat the 2 overlapping people when joining the three partial rows.
	Translation of question 3 from the self-assessment questionnaire on GA21's production: *After I finished drawing the 40 people, I realized that it could be done in other ways and that it was faster than drawing* [referring to calculating to find the correspondence between tables and people].

this item throughout the N-LOB cycles is influenced by the peer-assessments conducted by the students, as noted by Sanmartí (2010). In Table 4, we show how student GA21 evaluates her peer GA20's solution to Problem 1, indicating in questions 1 and 3 of the peer-assessment questionnaire that she does not understand the resolution process nor the meaning of the "speech bubbles" drawn by her classmate. Later, in Problem 3, we observe how student GA20 explains the process followed with much greater clarity and organization, thus applying item 24.4 of the N-LOB (*check that the solution makes sense, is clean and well written*) and anticipating the emergence of item 3.4.4 (*check that the solution makes sense, is clean well written and can be understood by another person*).

Regarding the base developed by the students in group B (Table 5), we can see that the group decided to separate phases 2 and 3 (planning and executing) in *N-LOB.3B*. In this case, items 3.2.1, 3.2.2, and 3.2.3, which contain planning verbs such as "reason" or "looking for", were categorised under phase 2 (planning), while items 3.3.1, 3.3.2, 3.3.3 and 3.3.4, which contain action verbs such as "calculate, draw, do or explain", were categorized

Table 4 Emergence of item 1.4.4 derived from the peer-assessment questionnaires and application of item 2.4.4 by student GA20 in Problem 3

Original production	Description and translation
	Translation of the student's text: *For 40 people, I drew 19 tables plus the ones on the sides* [referring to the people]. *For 143, I drew 70 tables in two rows plus the ones on the sides, and there is one spot left over.* *For 100, I drew 49 tables plus the ones on the sides* [referring to the people]. Description of student GA20's problem 1: The student draws rows of tables and adds "speech bubbles" with the numbers 40, 100, and 143 at different points in the drawing. He attempts a calculation to verify his result (bottom-left), but crosses it out because it does not correspond with the data in the drawing.
	Translation of questions 1 and 3 from the peer-assessment questionnaire by student GA21: *He drew 2 rows of 35 squares and from there counted, taking into account the two people at the ends. He marked where tables are needed with a kind of speech bubble.* *He barely explained anything, only the result and not the process. I don't understand the speech bubbles or why he repeated them.* *I think only he knows how to find the result easily because he didn't mark or indicate any calculation to make it understandable (he probably did it wrong).*
	Translation of the student's text: *I started from 10 numbers less after seeing how the numbers progressed. I wrote down the numbers one by one, and the numbers next to 100 and 1000 are 99, 97, 109 which sum to 305, and 999, 997, and 1009 which sum to 3005.* Description of student GA20's problem 3: The student draws the table starting from the numbers 91 and 991 respectively and he maintains the pattern shown in the problem's statement. He correctly finds the numbers around 100 and 1000 respectively and adds them at the bottom of the page.

under phase 3 (executing). This suggests that students in group B pay special emphasis on planning the strategy before implementing it, as expert problem solvers do (Schoenfeld, 1992). The division between the planning and executing phases can be observed in cases such as student GB3's work (Table 6). In Problem 3, we observe how student GB3 decides to extend the table up to number 110 in order to sum the numbers around 100 and extrapolate his result to number 1000 (simultaneously executing and planning his strategy). In contrast, in Problem 4, we see how the same student explains that, once he observes

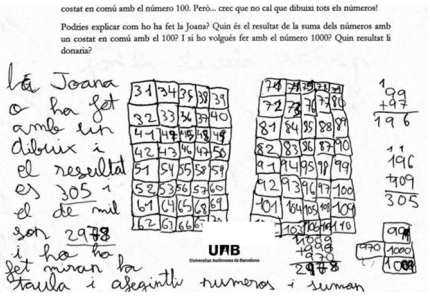
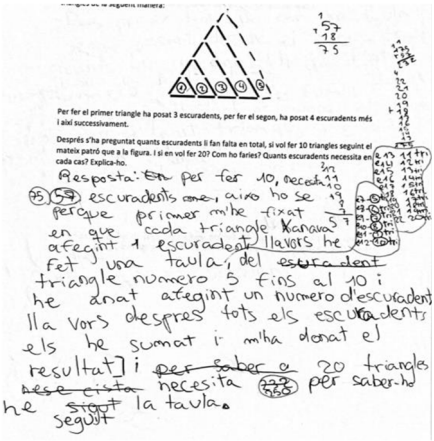
Table 5 Non-linear orientation bases—group B

	N-LOB.1B	N-LOB.2B	N-LOB.3B	
Understanding	1.1.1 Read as many times as necessary *(Regulatory Process)*	2.1.1 Read as many times as necessary *(Regulatory Process)*	3.1.1 I read as many times as necessary *(Regulatory Process)*	
	1.1.2 Remove from the statement what distracts me *(Regulatory Process)*	2.1.2 Remove from the statement what distracts me *(Regulatory Process)*	3.1.2 I remove from the statement what distracts me *(Regulatory Process)*	
	1.1.3 Take the important data *(Regulatory Process)*	2.1.3 Take the important data *(Regulatory Process)*	3.1.3 I select the important data (Regulatory processes)	
	1.1.4 Take the question *(Regulatory Process)*	2.1.4 Take the question *(Regulatory Process)*	3.1.4 I select the question *(Regulatory Process)*	
		2.1.5 Draw to understand *(Awareness Process of a Mathematical Tool)*	3.1.5 I draw to understand the pattern (if necessary) and look at the drawing in different ways (horizontal, vertical and whole) *(Awareness Process of Facts, Definitions and Mathematical Tools)*	
			3.1.6 I mentally organize the sheet *(Regulatory Process)*	
			3.1.7 I do trials *(Awareness and Evaluation Process)*	
Planning and executing	1.2.1 Analyse and look at the problem carefully *(Regulatory Process)*	2.2.1 Analyse and look at the problem carefully *(Regulatory Process)*	**Planning**	3.2.1 I review one step at a time before moving on to the next one *(Evaluation Process)*
	1.2.2 Reasoning what I have to do *(Regulatory Process)*	2.2.2 Reasoning what I have to do *(Regulatory Process)*		3.2.2 I reason what I need to do based on the trials I have done *(Regulatory Process)*
	1.3.1 Doing what the problem asks me to do *(Awareness Process)*	2.3.1 Doing what the problem asks me to do *(Awareness Process)*		3.2.3 I am looking for an easy or quick way to reach the solution *(Awareness Process)*
		2.3.2 Doing mental arithmetic *(Awareness Process of Algorithms)*	**Executing**	3.3.1 I calculate mentally *(Awareness Process of Algorithms)*
		2.3.3 Drawing to solve *(Awareness Process of Mathematical Tools)*		3.3.2 I enlarge the drawing to solve the problem (if needed) *(Awareness Process of Mathematical Tools)*
				3.3.3 I do calculations and explain them *(Awareness Process of Algorithms)*

Table 5 (continued)

	N-LOB.1B	N-LOB.2B	N-LOB.3B
			3.3.4 I explain the way I have chosen to solve the problem *(Awareness Process)*
Reviewing	1.4.1 Check the data in the statement *(Evaluation Process)*	2.4.1 Check the data in the statement *(Evaluation Process)*	3.4.1 I check the data in the statement *(Evaluation Process)*
	1.4.2 Check the calculations *(Evaluation Process of Algorithms)*	2.4.2 Check the calculations *(Evaluation Process of Algorithms)*	3.4.2 I check the calculations *(Evaluation Process of Algorithms)*
	1.4.3 Review the response *(Evaluation Process)*	2.4.3 Review the response *(Evaluation Process)*	3.4.3 I check the answer and mark it in another colour, with a box or circle *(Evaluation Process)*
	1.4.4 Re-read the statement and what I have done *(Regulatory Process)*	2.4.4 Re-read the statement and what I have done *(Regulatory Process)*	3.4.4 I re-read the statement and what I have done *(Regulatory Process)*
		2.4.5 Draw to review *(Evaluation Process of Mathematical Tools)*	3.4.5 I draw to check and corroborate the process and the solution. I check that the drawing is still the same as the one in the statement (Regulatory and Evaluation process of Mathematical Tools)
			3.4.6 I check that I have explained in detail what I am doing and thinking *(Evaluation Process)*

Table 6 GB3's production on Problem 3 (plans and executes simultaneously) versus GB3's production on Problem 4 (first plans and then executes)

Original production	Description and translation
	Translation of the student's text: *Joana made a drawing and the result is 305, and for one thousand it is 2978. She did this by looking at the table, adding numbers, and summing.* Description of student GB3's problem 3: The student extends the table up to number 110. Subsequently, he observes that the numbers around 100 are 99, 97, and 109, which he adds on the right side of the page (305). Finally, he extrapolates that around the number 1000 the numbers will be 999, 970, and 1009, and adds them, obtaining 2978. He does not notice that the number 970 should be 997 and therefore obtains an incorrect result.
	Translation of the student's text: *Answer: To make 10* [triangles], *I need 75 chopsticks. I know this because I first noticed that for each triangle, I was adding one chopstick. So, I made a table from triangle number 5 to 10 and kept adding chopsticks. Then, I added all the chopsticks and got the result. For 20 triangles, I need 250. To find this out, I followed the table.* Description of student GB3's problem 4: The student observes that from one triangle to the next, the number of chopsticks added is equal to the previous amount plus one and considers making a table to explore this growth (planning). On the right side of the page, we observe the complete table (execution), and additionally, we see how he adds an arrow linking the table to the exact part of the text where he explains his execution.

the pattern (an increase of one chopstick per triangle), he considers creating a table to find the total number of chopsticks, thereby separating the planning phase from the execution phase.

In addition, we observe that *N-LOB*.1B and *N-LOB*.2B are written with infinitive verbs such as "review" or "draw", whereas *N-LOB*.3B is written with indicative mood verbs and in the first person. This decision was made by the students during the discussion to modify the N-LOB throughout the problems. We interpret this as an indication that, as students used the base to support their own metacognition, they needed to engage with it in the first person rather than the third person. This shift in verb tense suggests an increase in the importance of the N-LOB, as noted by García and Sanmartí (1998).

Similar to group A, students in group B add metacognitive awareness processes in the "understanding" phase. They also note regulatory processes in the "planning" and "reviewing" phases. Again, this reflects the nonlinearity of the metacognitive processes that students use during problem solving.

Analysing the first phase, "understanding" the problem, we see that students add item 2.1.5, which turns into item 3.1.5. In this last item, students note the *analysis of the drawing from different perspectives* to identify regularities among the elements. This introduces an awareness process of facts, definitions and mathematical tools that is highly relevant to solving pattern problems. This awareness process provides evidence of planning strategies that are more efficient than simply continuing the representation of the problem statement (Torregrosa et al., 2020). Within this first phase, *N-LOB.3B* also includes items 3.1.6 and 3.1.7, which relate to *organising the sheet* and *testing ideas before proceeding with the final design of the plan*. These regulatory and awareness items come from the peer-assessment questionnaires (as we can see in Table 7), as they allow students to understand when their peers are testing ideas and when they are starting the final solution.

Continuing with the case of student GB3 presented above, we observe that when assessing his peer GB4 during the peer-assessment of problems 1 and 3, he states in question 1 that he does not understand what his classmate initially thought in order to solve the problems (Table 7). In contrast, in Problem 4, he states that he does understand what his peer has done, as she conducts trials at the top of the page (including arrows and clarifications in a different colour) and explains her reasoning at the bottom.

Finally, we observe a shifting vocabulary between *N-LOB.2B* and *N-LOB.3B*. Students move from using more colloquial verbs like "take or do" to using more mathematically appropriate verbs like "remove or select".

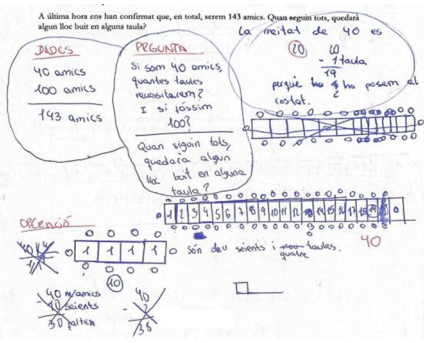
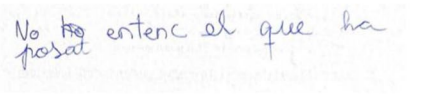
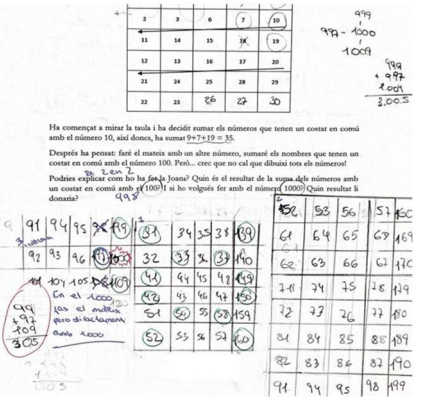
When analysing the "planning" and "executing" phases, we see a significant metacognitive evolution, as they have moved from more general processes in *N-LOB.1B* (with items such as 1.3.1 "*Doing what the problem asks me to do*") to processes and skills specifically tailored to pattern problems in *N-LOB.3B* (with items such as 3.2.2 or 3.2.3, which suggest *creating a strategy based on trials* and *finding quick or easy ways that are not always limited to continuing the representation of the problem statement*). In terms of mathematical skills, items such as 2.3.3, which suggests *extending the drawing as a solution strategy*, are adapted to emphasise *choosing the most efficient strategy* (item 3.3.2).

When analysing the "reviewing" phase, we note that students in *N-LOB.2B* add item 2.4.5, which refers to *using drawings* not only as an awareness process, but also as an evaluation or review process. This shows how a skill related to a mathematical tool can be present in more than one type of metacognitive process. As we pointed out in the "understanding" phase, we find that in the "reviewing" phase, students also modify items such as 3.4.3 and 3.4.5 by indicating that *the answer should be marked in a different colour* (as can be observed in student GB4's problem 4 in Table 7) and that the representation of the pattern should remain the same as in the one in the statement. Again, these evaluation processes emerge from the peer-evaluation questionnaires completed by the students, which emphasize the need to make their solution understandable to their peers.

Regarding the N-LOB developed by the students in group C (Table 8), we observe that the group has chosen not to separate phases 2 and 3 (planning and executing), similar to the students in group A. This suggests that the group perceives the creation and execution of the plan as a single interrelated process.

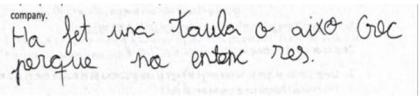
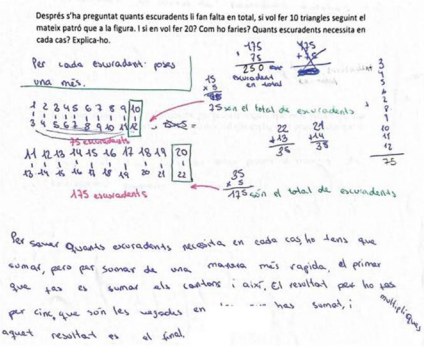
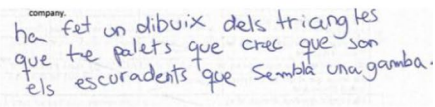
Overall, *N-LOB.1C* is quite general and lacks precision. This is evident from the limited presence of mathematical skills and the lack of detail in the stages, with items such as 1.3.1, "*I do what I thought*". On the other hand, in *N-LOB.3C*, we observe the emergence of skills mainly related to mathematical tools, such as explicit references to the *continuation of the pattern* (item 3.1.5). In this sense, the base becomes increasingly specific and oriented towards pattern problems.

Table 7 Student GB4's Solutions to problems 1, 3, and 4 accompanied by peer-assessment conducted by Student GB3

Original production	**Description and translation**
GB4's solution to problem 1  	Description and translation of student GB4's problem 1: The student writes the data (first circle) and the question (second circle) exactly as they appear in the problem statement. Subsequently, she performs a calculation (20 - 1 = 19), writing "*half of 40 is 20, and it's 19 because we put 2 on the side*," referring to removing one table because two people sit at the end of the same table. At the bottom, she draws the 19 tables to verify her reasoning and makes several attempts with calculations to find the number of tables for 100 people, but she does not succeed (crossing out her attempts). Finally, she writes "*40*" in red, although it is unclear where she obtained the number 40 from.
Peer-assessment conducted by GB3 of GB4's problem 1 (Question 1)  	Translation of GB3's peer assessment of student GB4: *I don't understand what she has written.*
GB4's solution to problem 3  	Description and translation of student GB4's problem 3: The student extends the table from number 30 to 60 (table in the center of the page). Subsequently, she extends the table from 52 to 98 (table on the right), adding a final column to the right of this table to approach 200 while maintaining the pattern (160, 169, 170, 179…). Finally, on the left side of the page, she adds the continuation of the table from 91 to 109 and sums the numbers 99 + 97 + 198 = 305, writing "*For 1000 you do the same but up to 1000.*" We observe that at the top of the page she sums 999 + 997 + 1009 = 3005.

When analysing the first stage, "understanding", we find that it is the stage that undergoes the least change. In *N-LOB.3C*, students add item 3.1.5, which refers to *drawing the data and the pattern to visualize the problem well*. We believe that this item arises from the need to find relationships between the pictorial elements of the problem statement to

Table 7 (continued)

Peer-assessment conducted by GB3 of GB4's problem 3 (Question 1)  	Translation of GB3's peer-assessment of student GB4: *She made a table, or at least I think so, because I don't understand anything.*
GB4's solution to problem 4  	Translation of the student's text: *To find out how many chopsticks you need in each case, you have to add them all, but to sum more quickly, the first thing you do is add the corners and so on* [we observe how in the top table she relates the numbers from the ends toward the center using Gauss's method to sum 3+4+5+6+7+8+9+10+11+12, thus obtaining the sum for the first and tenth triangles, which equals 15]. *You multiply the result by 5, which is the number of times you need to sum, and this gives the final result.* Description of student GB4's problem 4: The student creates two tables showing the correspondence between the triangle number and the chopsticks added (in the first table up to triangle 10 and in the second table up to triangle 20). She highlights these two triangles with a green rectangle and adds clarifications in red: "*75 chopsticks*" and "*175 chopsticks*." She links the calculations she performed (sum of chopsticks up to 10 and 20 triangles, respectively) with a red arrow pointing to each table and writes at the end of the calculations in green: "*75 is the total number of chopsticks*" and "*175 is the total number of chopsticks*."
Peer-assessment conducted by GB3 of GB4's problem 4 (Question 1)  	Translation of peer-assessment by GB3 of GB4.4: *She made a drawing of the triangles* [referring to the table] *that have sticks which are the chopsticks, and it looks like a shrimp* [referring to the shape of the drawing].

apply more efficient solution strategies than simply extending the drawing (as can be seen in Table 9).

In the first column of Table 9, student GC18 draws the path followed by the arrows in Problem 3 to understand how to complete the table. Subsequently, his peer evaluator GC17, who had not marked any path in Problem 3 (second column of Table 9), applied this visualization in the drawing of the problem 4 statement (third column of Table 9).

In fact, in *N-LOB.3C*, we find item 3.3.4 in which students state, "*If I don't come up with a strategy, I draw the pattern*". In this regard, we believe that students understand that pictorially extending the pattern is an efficient process for understanding and planning, but inefficient for solving the problem. This fact was highlighted by several students

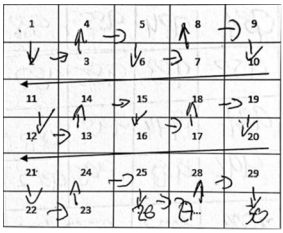
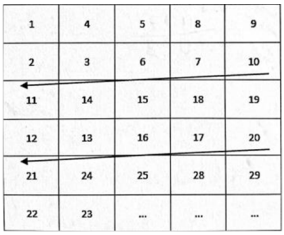
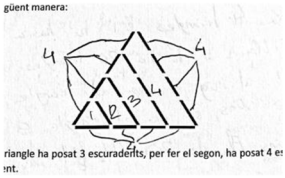
Table 8 Non-linear orientation bases—Group C

	N-LOB.1C	N-LOB.2C	N-LOB.3C
Understanding	1.1.1 Read many times *(Regulatory Process)*	2.1.1 Read many times *(Regulatory Process)*	3.1.1 I read as many times as necessary *(Regulatory Process)*
	1.1.2 I remove from the statement what distracts me *(Regulatory Process)*	2.1.2 I remove from the statement what distracts me *(Regulatory Process)*	3.1.2 I remove from the statement what distracts me *(Regulatory Process)*
	1.1.3 I take the important data *(Regulatory Process)*	2.1.3 I select the important data *(Regulatory Process)*	3.1.3. I select the important data *(Regulatory Process)*
	1.1.4 I take the question *(Regulatory Process)*	2.1.4 I select the question *(Regulatory Process)*	3.1.4 I select the question *(Regulatory Process)*
			3.1.5 I draw the data and the pattern to visualize the problem well *(Awareness Processes of Facts, Definitions and Mathematical Tools)*
Planning and executing	1.2.1 I analyse the problem *(Regulatory Process)*	2.2.1 I analyse and look carefully at the problem *(Regulatory Process)*	3.2.1 I analyse and think about the strategy I will follow *(Regulatory Process)*
	1.2.2 I think about what I am going to do *(Regulatory Process)*	2.2.2 I think about what I have to do *(Regulatory Process)*	3.3.1 I do what I am asked to do with the strategy I have followed *(Awareness Process)*
	1.3.1 I do what I thought *(Awareness Process)*	2.3.1 I do what the problem asks me to do *(Awareness Process)*	3.3.2 Change of strategy if the one I have used does not work for me *(Awareness and Evaluation process)*
		2.3.2 I count mentally, and I do written calculations *(Awareness Process of Algorithms)*	3.3.3 I count mentally, and I do written calculations *(Awareness Process of Algorithms)*
			3.3.4 If I don't come up with a strategy, I draw the pattern *(Awareness and Evaluation process of Mathematical Tools, Facts and Definitions)*

Table 8 (continued)

	N-LOB.1C	N-LOB.2C	N-LOB.3C
Reviewing	1.4.1 I check what I was asked for *(Evaluation Process)*	2.4.1 I check what I was asked for *(Evaluation Process)*	3.4.1 I check the data in the statement and in the drawing *(Evaluation Process of Mathematical Tools)*
	1.4.2 I review the calculations *(Evaluation Process of Algorithms)*	2.4.2 I review the calculations *(Evaluation Process of Algorithms)*	3.4.2 I review the calculations *(Evaluation Process of Algorithms)*
	1.4.3 I check the answer *(Evaluation Process)*	2.4.3 I check the answer *(Evaluation Process)*	3.4.3 I check the answer *(Evaluation Process)*
	1.4.4 I read everything again (the statement and what I have done) *(Evaluation Process)*	2.4.4 I read everything again (the statement and what I have done) *(Evaluation Process)*	3.4.4 I read everything again (the statement and what I have done) *(Evaluation Process)*
			3.4.5 Drawing to check that my calculations match the drawing of the statement *(Regulatory and Evaluation process of Mathematical Tools)*

Table 9 Application of item 3.1.5 "Drawing the data and the pattern to visualize the problem well" in student GC17's problem 4

Problem 3 GC18	Problem 3 GC17	Problem 4 GC17
		

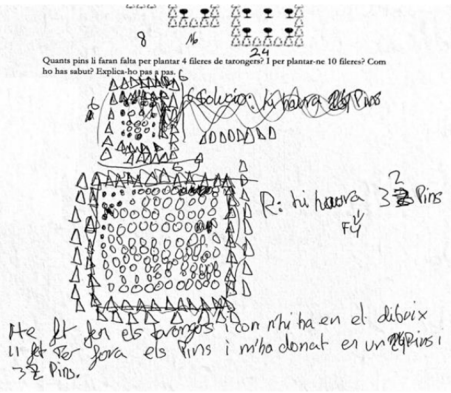
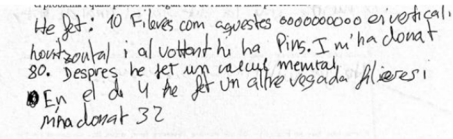
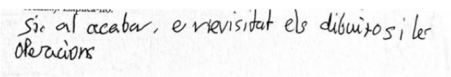
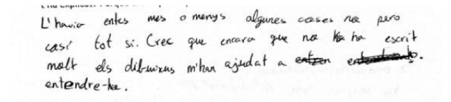
in the group sharing, as they observed that enlarging the drawing to the desired step was very time-consuming and prone to error. For this reason, in the final discussion that led to *N-LOB.3C*, they decided to add this item.

In analysing the planning and execution phases, *N-LOB.2C* includes modifications to almost all of the items listed in *N-LOB.1C*. These changes may seem minor, but they are significant because they relate to processes that need to be made more concrete so that students can apply them to the next problem. In this sense, item 2.2.1 emphasises the need to "*analyse and look carefully*" at the problem (referring to the pictorial representation), and item 2.3.1 refers to "*doing what the problem asks"* (clarifying item 1.3.1, which could give the impression of finding the pattern without considering what the problem specifically asks). In addition, *N-LOB.2C* introduces item 2.3.2, which refers to *doing written calculations*. As with the students in group A, we believe that the inclusion of this item is a result of the peer-assessment questionnaires (as shown in Table 10). In this case, we can observe that student GC18, presented in the previous example, performs a mental calculation to solve problem 2. He states this mental calculation in question 1 of his self-assessment questionnaire and in question 4 regarding the review of the process. His peer evaluator, GC17, indicates in the peer-assessment questionnaire for problem 2 that the drawings helped her, but there are things she does not understand (referring to the calculations that student GC18 performed mentally and did not write down).

Finally, it is worth noting that *N-LOB.3C* shows a significant development in the awareness and evaluation of metacognitive processes. We note the appearance of two items (3.3.2 and 3.3.4) that refer to *changing strategies and drawing the pattern as a last resort when no arithmetic or algebraic methods are found*. In this respect, we agree with Villalonga and Deulofeu (2017) that such processes are key to avoid potential blocks during problem solving.

Finally, when analysing the "reviewing" phase, we observe the evolution from item 1.4.1 ("*I check what I was asked for*") to item 3.4.1 ("*I check the data in the statement and in the drawing*"). We consider that this clarification reflects the need not only to identify the question to be answered, but also to review the data available, emphasising that the pictorial representation of the statement is a valuable source of data. Finally, students add item 3.4.5 to *N-LOB.3C*, which reinforces the idea that they should not only check their calculations, but also *make sure that they match the data provided by the problem representation*. As Pólya (1945) points out, the emergence of such processes, which combine the evaluation and regulation of arithmetic strategies, indicates a good mastery of problem solving heuristics in pattern problems.

Table 10 Emergence of item 2.3.2 "I count mentally, and I do written calculations" in the work of the student pair GC17 and GC18

Original production	Description and translation
GC18's solution to problem 2  	Description and translation of GC18's problem 2: The student writes the numbers 8, 16, and 24 below the drawings in the problem statement. Next, he draws the 4 rows of orange trees surrounded by pine trees and the 10 rows of orange trees surrounded by pine trees. Then, he writes in the right margin, "*There will be 4 rows × 32 pines*" (referring to 4 rows). At the bottom, he writes, "*I drew the orange trees as in the picture and placed the pines outside, and I got 32 pines*."
GC18's self-assessment for problem 2. Responses to questions 1 and 4.   	Description and translation of GC18's self-assessment for problem 2: *I made 10 rows like these* [he draws several circles in a row] *vertically and horizontally, with pines around them. And it gave me 80* [although the number 80 does not appear in his work]. *Then I did a mental calculation. For the 4 rows, I made rows again and I got 32.* *Yes, after finishing, I reviewed the drawings and the calculations.*
Peer assessment by student GC17 (question 1) of student GC18's problem 2  	Description and translation of peer-assessment by student GC17 (question 1): *I understood it more or less. Some things I didn't, but most of it I did. I think that although not much was written, the drawings helped me understand it.*

7 Discussion

The characterisation of mathematical skills and metacognitive processes allowed us to make some general observations about self-regulation in problem solving. First, we observed that during the "understanding" phase, all three groups of participating students identified regulatory processes that are not usually associated with skills specific to pattern problems. This leads us to suggest that metacognitive regulatory processes,

i.e., those that allow us to understand a problem prior to solving it, are more general than awareness or evaluation processes.

Secondly, during the "planning" and "executing" phases, awareness and evaluation processes have emerged that are fully interrelated at different moments and do not necessarily have to be carried out chronologically. The nonlinearity of the Non-Linear Orientation Base addresses a limitation identified by Villalonga and Deulofeu (2017), as well as by Villalonga and Andrews (2018), who mentioned the linearity of the Orientation Base of Action as a limitation when applied during problem solving. In this sense, the flexibility of the Non-Linear Orientation Base allows students to move more fluidly between the phases of problem solving without having to follow a fixed step-by-step sequence.

Thirdly, we have observed that as students move through new cycles of problem solving, group sharing, evaluation, and discussion, mathematical skills related to the development of more specific strategies for pattern problems—and thus closer to generalisation—emerge. In this respect, *N-LOB.3B* and *N-LOB.3C* show that extending the drawing from the problem statement as a solution strategy is used only when students are unable to find more efficient strategies. Evidently, the fact that the students solved four problems of increasing difficulty plays a significant role in the emergence of these specific and efficient strategies. However, we consider that the use of the Non-Linear Orientation Base has also facilitated awareness of various metacognitive processes that help prevent "lack of progress" (Goos et al., 2000). Furthermore, as anticipated in the theoretical framework, we have observed that the more cycles carried out, the more specific and detailed the base becomes. In conclusion, we note that the implementation of 4 cycles of creation and modification of a Non-Linear Orientation Base seems to be sufficient to promote the emergence of awareness and evaluation processes that help to evaluate the use of certain mathematical strategies.

8 Conclusions

This research contributes to the study of metacognition in problem solving through the creation and use of a Non-Linear Orientation Base (N-LOB). This instrument was found to be a valuable tool for promoting metacognitive self-regulation in mathematics education, particularly in helping students to solve non-routine problems that require flexibility. The study concludes that the use of an N-LOB has some potential for improving students' ability to plan, control and evaluate their problem-solving strategies. The iterative cycles of problem solving, group sharing, self-assessment, peer-assessment and team discussion among students foster a deeper understanding of metacognitive processes. These cycles not only encourage self-regulation, but also provide teachers and researchers with valuable insights into how students approach and adapt their problem-solving strategies. In addition, it appears that using a N-LOB can prevent cognitive blocking (Goos et al., 2000) as shown in Table 3 for group A, in which the student changes the strategy during the problem solving process.

Returning to the theoretical framework presented, we agree with Papaleontiou-Louca (2008) in emphasising that metacognitive self-regulation can be developed over time through activities and tools that promote regulation, awareness, and evaluation processes. In this sense, mathematical pattern problems have proven to be an ideal

type of activity for manifesting metacognitive processes. Regarding the N-LOB, we consider that it may address the limitation identified by An and Cao (2014) regarding the low validity of certain metacognitive self-regulation instruments, particularly those designed by researchers rather than by the problem solvers themselves.

In addition, we agree with Gama (2001) that sustained reflection over time appears to be one of the key factors in improving self-regulation. However, as with self-monitoring questionnaires (Goos et al., 2000), self-regulation tools alone do not guarantee complete success. This leads us to suggest that the influence of the N-LOB on self-regulation may be closely related to the type of problems posed, as well as to the use of self- and peer-assessment questionnaires. This brings us to one of the limitations of the study: the fact that the metacognitive processes identified in N-LOB.3 are more complex and deeper than those identified in N-LOB.1 could be due to several factors beyond the creation and development of the instrument itself.

The overall results of the study suggest that the N-LOB could be adapted to other areas of mathematics beyond pattern problems, as its structure is flexible enough to accommodate different types of problems and cognitive demands. It would be interesting to explore how students adapt an initial N-LOB to a different mathematical context and how much they need the tool in their process of mathematical self-regulation. In this regard, we consider that adapting the N-LOB to other mathematical contents could follow a scheme similar to Fig. 4, since for each type of mathematical content we will find common processes and skills as well as processes and skills specifically linked to the mathematical content.

The study opens the way for future research on the use of the N-LOB in other mathematical topics or processes, such as modelling. In addition, it raises the possibility of using the N-LOB for problem posing activities (Cai & Hwang, 2023), which could further enhance students' metacognitive skills.

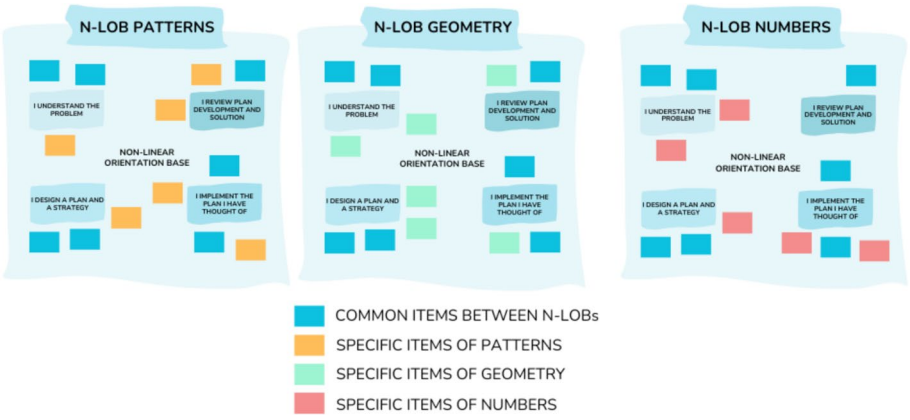


Fig. 4 Possible evolution of a non-linear orientation base

Appendix



Fig. 5 Schematic representation of the study methodology

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Data availability The data that support the findings of this study are available from the corresponding author upon request.

Declarations

Conflict of interest The authors declare no competing interests.

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