FPAT – A Framework for Facilitating the Teaching and Learning of Using Fermi Problems Originating in Mathematics Education Research

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Enrico Fermi is remembered for his many contributions to theoretical and experimental physics, but from an educational point of view he also popularized the use of the kind of questions we now call "Fermi problems" (or "Fermi questions"[1]). Fermi problems (FPs) are back-of-envelope problems that arose from the need to make order of magnitude calculations, and Fermi himself used them in his own research and physics classes. The procedure proposed by Fermi was to break the original problem down into simpler sub-problems, solve these by making reasonable estimates and educated guesses, and thereby reach a solution to the original question. In the literature, this way of working is known as the "Fermi (estimates) method."

Although originating in physics, the use of FPs has spread to different subjects and contexts, with multiple purposes across various educational levels.[2] However, educational research focusing on FPs has been relatively scarce. In this article we will highlight some of the recent research developments on the use of FPs in mathematics education. Our aim is to present a framework for facilitating the use of FPs with students, and to call for further research focusing on the educational use of FPs within and across subject disciplines.

Fermi problems in physics education

Rather than providing a comprehensive review of the literature on FPs in physics education (for a general review, see Ärlebäck and Albarracín[2]), we seek to highlight some of the research findings. For example, studying the correlation between first-year university students learning the Fermi estimates method and their examination results, Robinson showed that students can successfully develop this problem-solving competence to solve FPs effectively, but that such a skill does not come easily for all students.[3] Barahmeh et al. also showed this in the context of teaching ninth-grade students physics and illustrated how the students' skills developed in terms of making measurement predictions, taking measurements, and improving the productive use of measurements.[4] Both Robinson and Barahmeh et al. stress the value of the Fermi method for allowing students to think freely without solely relying on predetermined formulas for solving tasks, and for promoting discussions of different solutions. Cordry on the other hand used the Fermi estimation method to raise critical environmental issues concerning the world's population and the limited resources of planet Earth. Students were generally surprised by the results of engaging in this problem—especially when they realized that the current global human population exceeds what they had just calculated to be feasibly sustainable.[5] The situation that arose allowed for a discussion about the limitations of the method, such as some quantities being difficult to estimate and the values of other quantities being inferred by old habits or taken for granted. Similar results regarding raising critical awareness of one's beliefs and understandings are presented by Morgan, who used FPs based on the Drake equation and a pre/posttest strategy to show the change in students' opinions and beliefs about the existence of extraterrestrial life.[6] Together, these studies point to the potential and use of FPs to contrast and discuss different solutions and opinions, and to impact beliefs on other important issues such as climate change.

Fermi problems in mathematics education

There has been a recent line of research in mathematical education exploring the possibilities of using FPs in the classroom in the context of mathematical modeling. Mathematical modeling activities are in many respects similar to the practices of scientific research, but the focus is on the mathematical nature of the models used and developed and not on the phenomenon studied per se. However, many of the findings from this research are arguably transferable to the classrooms of other sciences if adapted and interpreted adequately with respect to the discipline in question. Importantly, FPs can be used for different pedagogical purposes at practically all educational levels. For early primary students, they can be used as a basis for classroom projects such as devising a strategy to compare the populations of a city and a town.[7] With younger students, it might be necessary for the teacher to provide (part of) an outline of the necessary estimates needed. For older primary students, however, Peter-Koop found that students are fully capable of generating their own mathematical models when, for example, estimating the number of cars in a motorway traffic jam. Indeed, the students solved the problem in myriad ways while developing new mathematical knowledge in order to work out their solutions.[8] A positive effect on the development of primary students' general modeling skills has also been shown, as FPs facilitate awareness of the different phases of problem solving in context as well as the development of the modeling sub-competences of simplifying, mathematizing, interpreting, and explaining real phenomena.[9]

At the lower secondary level, some students can develop considerably more complex models than primary students,[10] whereas upper secondary students can generally use their more extensive knowledge of the real world to generate even more sophisticated models.[11] One of the difficulties students experience is determining the key variables of a problem. Here, engaging students in collaborative work or in carefully designed sequences of problems using different real contexts has proven to promote the development and adaptation of increasingly complex models.[12] In the latter approach, the key has been to base the sequence on a particular concept, such as density, and to propose several problems where density is a

central idea, allowing students to develop their own models. In addition, when working with sequences of problems, secondary students develop their flexibility as problem solvers and their ability to choose and use appropriate strategies in different problem contexts.[13]

At the college level, FPs can have a more specific role and support students in developing more specific modeling competences. However, even for university students the modeling needed to solve an FP is complex, requiring them to regulate their modeling processes by monitoring how their immediate goals or subgoals align and relate to the problem statement.[14] In addition, FPs have also been seen to encourage students to validate their problem-solving processes and results in modeling activities.[15]

Recent developments: Fermi problem activity diagram

Focusing on the research in mathematics education and the other STEM disciplines, our review of the literature on FPs[1] has allowed us to identify four types of mathematical activities that are used to achieve the required numerical values of quantities to be able to provide a solution and an answer to the problem in question. These four activities are:

1) ***guesstimation*** (providing a fast answer as in traditional FP solving);
2) ***measurement and experimentation*** (of relevant values, in the lab or the field and using tools);
3) ***looking up data*** (in official databases or online newspapers); and
4) ***polling or statistical data collection*** (using surveys and questionaries, etc.).

A *Fermi Problem Activity Template (FPAT)* is a characterization that focuses on the structure of the solution of an FP in terms of the four different types of activities outlined above.[16] These templates show the sub-problems and variables that need to be considered, the type of activity that allows each partial result to be achieved, and the operations that combine these to arrive at the final result. In the FPAT, each activity is represented by a specific symbol or shape according to the following scheme: guesstimation (ellipse), experimentation (trapezoid), looking up data (rectangle), and polling or statistical data collection (hexagon). Figure 1 presents an FPAT that describes a possible way to estimate the number of rolls of toilet paper needed in a school in a school year.

To determine how many toilet visits a person makes each day, a statistical survey or data collection could be conducted (hexagon). Due to the potentially delicate nature of the amount of toilet paper used, guesstimation could be used to estimate the average toilet paper use per toilet visit (ellipse). To find the length of the paper on a toilet paper roll, one can engage in trying to calculate the total length using geometrical arguments or, as suggested in Fig. 1, conduct an investigation involving either unrolling the paper roll and measuring the length or some weighing process (trapezoid). It is suggested that the last two pieces of data needed to complete the calculation, i.e., the number of school days in a year and the number people in the school, are looked up (rectangles).

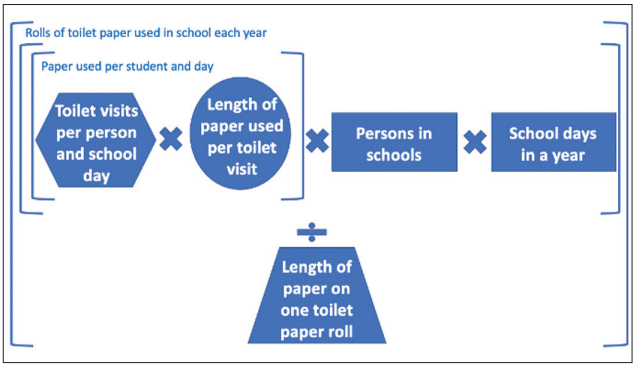

Fig. 1. An FPAT for the toilet paper roll task.

The FPAT graphical representation of the four potential activities (guesstimation, experimentation, looking up data, and polling or statistical data collection) provides a quick overview of the envisioned and potential solutions to an FP, as well as a way to visualize and represent an *a posteriori* analysis of students' work. Hence, an FPAT characterization of a task can have a variety of uses. From the point of view of activity design, the FPAT allows the teacher to anticipate what to expect in terms of which different problem-solving routes the students might take and what to prepare in class. In this way, the FPAT reduces some of the challenging aspects associated with the openness of using FPs in class. During classroom work, FPATs can be used to show learners the solution methods used and to help them become aware of the procedures used and how to combine them to generate results. In group discussions in which results are contrasted, FPATs can help to show the weaknesses of a specific approach or to compare structurally different resolutions.

However, given the differences in epistemologies, learning goals, and practices of teaching and learning mathematics on the one hand, and physics on the other, what would an FPAT in a physics education setting look like? For example, in physics the activities of *experimentation* and *polling* are fundamentally intertwined and hence it is probably not as productive to separate the two activity types in a physics context. In addition, well-established physical laws and defined quantities play a central role in physics and potentially provide at least a partial prestructuring of some of the aspects involved in solving FPs in physics education settings. We illustrate this latter point using an example of an FP combining two FPs discussed in Ärlebäck[11] and Weinstein:[17] *How much power does an average student use climbing the stairs to the observation deck in the Empire State Building?*

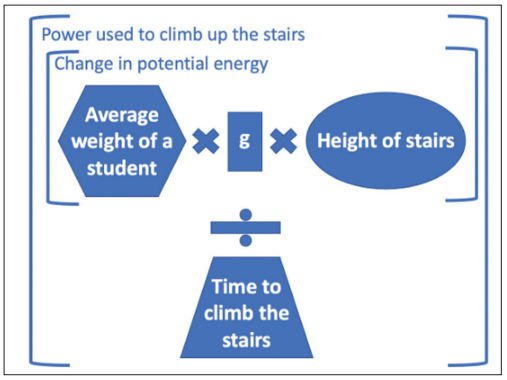

Fig. 2. The FPAT describing the structure of the Empire State Building power task.

This FP is prestructured by the definition of power (P), understood

and measured in watts or joules per second (1 W = 1 J/s), which can be determined by estimating the change in potential energy during a given amount of time; if the mass m in kg is displaced vertically h meters in t seconds, then the power used is $P = mgh/t$ (g being the acceleration due to Earth's gravity). Hence, the corresponding FPAT for this problem has the structure illustrated in Fig. 2.

To solve this FP, students in a class measure their own weights in order to establish a representative measure of central tendency (mean, median, or mode, for example) of an average student's weight using statistical analysis (hexagon). To determine the height of the stairs, the distance between the ground floor and the observatory floor, a guesstimation approach could be used by students sharing and discussing their own personal experiences of tall buildings (ellipse). Finally, to find the estimated time it takes to climb all the stairs, the students can conduct an experiment using the stairs in the school (trapezoid) and then use proportional reasoning to scale up to the estimated height of the stairs in the Empire State Building. In addition, if an effort is made to keep track of the uncertainty and variation in the derived estimated values (using measures of spread for the weight and a fixed percentage error margin for the height and the time, for example), an error and propagation of error analysis based on the structure of the FPAT could be calculated, resulting in an interval capturing reasonable answers to the FP.

In this example, an upper bound can be established using the official data from the Empire State Building Run-Up-,in which runners cover the 1576 steps and 86 floors up to the (lower) observation deck situated 320 meters above the ground floor. Using the record time for men (9 minutes and 33 seconds) and women (11 minutes and 23 seconds),[18] together with an estimate of a runner's weight, an upper bound can be calculated.

Final remarks

With this contribution, we intend to show that there is a plethora of research showing the potential of FPs as classroom tools that transcend the initial and traditional use of Fermi. However, many aspects of the use of FPs in educational settings are still unexplored, from both in-discipline and cross-discipline contexts. We are convinced of the potential and multiple benefits of using FPs in classrooms, and that developing theoretical and practical tools to facilitate the use and further uptake of FPs would be a fruitful endeavor. In a globalized world where large and hard-to-grasp numbers figure in the public discourse and decision making that directly affect us all, we believe that FPs can play a fundamental role in promoting the skills needed to understand the world better, by focusing on creative ways of thinking, developing critical thinking, and supporting decision making. In particular, we believe that future research could focus productively on empirical studies investigating similarities and differences related to the use of FPs in different disciplines. Connected to the FPAT framework, interesting questions to look into include *How are similarities and differences manifested in the FPAT framework in the teaching and learning of mathematics compared with the teaching and learning of physics*? and *What characterizes an FPAT representation of students' work in introductory classes compared to more advanced classes?*

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