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ORIGINAL PAPER

Teacher education for mathematical modelling: exploring the experiences of secondary school teachers in two courses

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Abstract
This paper focuses on in-service teacher education as a crucial agent in driving educational paradigm shifts for the teaching and learning of mathematical modelling. Our paper presents empirical research based on two case studies, examining training courses for in-service secondary school mathematics teachers. The first course integrates different modelling approaches, while the second one is based on the anthropological theory of the didactic and uses the proposal of the study and research paths for teacher education. Both experiences conclude with teachers working on the design of modelling tasks to be implemented in secondary schools. Both courses are analysed through the lenses of how teachers progress on the collective identification of conditions and constraints for modelling, when teachers deal with the design of modelling tasks and analyse their implementations. First, the results show the kind of modelling tasks teachers design and the level of disciplinary specificity at which they plan modelling to be implemented. Second, we analyse how teachers address the "ecological analysis", that is, the identification of conditions and constraints that facilitate or hinder the integration of modelling in secondary school classrooms after their experiences.

Keywords Mathematical modelling · Teacher education · Ecology · Modelling cycle · Study and research paths

1 Introduction

Mathematical modelling and how to sustain its long-term dissemination in schools has become a significant research topic in the last decades (Cevikbas et al., 2021). Currently, there is a consensus that modelling plays an important role in mathematics education to improve students' mathematical skills and promote responsible citizenship (Maaß et al., 2019; Schukajlow et al., 2018).

Undoubtedly, teachers play a crucial role in this educational shift (Swan et al., 2013), as they are key agents in turning modelling into a widespread practice in classrooms. However, teachers are part of complex systems, where they might address the existence of many difficulties for mathematical modelling to be short and long-term implemented and self-sustained in school classrooms. This paper aims to present an empirical research with two in-service teacher education courses, both addressing the topic of how to teach mathematical modelling, as an opportunity to collectively (teachers, educators and researchers) progress on the identification of constraints for modelling.

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2 Literature review and theoretical considerations

2.1 Advances in the research field of mathematical modelling

The field of research on mathematical modelling has made significant advances over the past decades. The promotion of modelling competencies has gained global recognition as a pivotal goal for mathematics education (Kaiser, 2020). The modelling research community has attempted to clarify and connect different frameworks for designing and analyzing mathematical modelling activities (Kaiser & Sriraman,

2006). When addressing challenges related to the teaching and learning of mathematical modelling, diverse conceptions of modelling emerge. For instance, the *modelling cycle and competency-based* approach (Blum, 1991, 2015) offers the prevailing interpretation of modelling in this field and appears to be particularly valuable for analysing the *cognitive processes* undertaken by students and teachers (Borromeo Ferri, 2018). The *socio-critical modelling* perspective aims to develop mathematical modelling skills to critically understand the surrounding world (Barbosa, 2006). The *models and modelling perspective* provides a framework where modelling is described as a sequence of model development, involving different types of activities: model creation, exploration and adaptation (Lesh & Doerr, 2003). Within the framework of the *anthropological theory of the didactic*, modelling is interpreted as a process of reconstructing and articulating mathematical organizations of increasing complexity. In this framework, the s*tudy and research paths* (SRPs) are proposed as inquiry-based instructional devices for the teaching of mathematics as a modelling tool (Bosch, 2018).

Within this research field, several studies point in the same direction concerning how important the role of the teacher is in the modelling processes and the great impact teachers have on students' modelling processes. Some contributions have addressed more specifically the challenge of teacher education with two main aims. On the one hand, various investigations have analysed the proposal of teacher education programmes for modelling for pre-service (i.e., Greefrath et al., 2022; Kaiser et al., 2010; Orey & Rosa, 2018) and in-service teachers (i.e., Blomhøj & Kjeldsen, 2006; Jessen & Rasmussen, 2020). To enhance teachers' professional knowledge and competences for the teaching and learning of modelling, they consider different formats and contexts based on different theoretical approaches to modelling.

On the other hand, some studies have identified different obstacles teachers find when implementing mathematical modelling. As pointed out by Greefrath and Vorhölter (2016, p. 28), "teachers not only have to be convinced of the usefulness of mathematical modelling; rather, they have to overcome suspected obstacles." Barquero et al. (2018) indicate that previous research mostly focused on an *individual* approach to the conditions and constraints found by teachers and/or students in their scope of action. However, constraints appear at different levels of generality. For instance, Burkhardt (2018) discusses the existence of barriers coming from the systemic inertia of curricula, or from the limited professional development of teachers. Dorier and García (2013) argue that teachers are part of a complex system with its dynamics and rules, affecting and shaping their actions. It is thus essential that in-service teacher education, as a complement to a more individual approach, also embraces an *institutional* approach to focus on the conditions and constraints offered by the different institutions to facilitate the effective dissemination of modelling.

How joint progress should be made when proposing teacher education programmes, and how these programmes can advance on the identification of existing *constraints* for modelling, are questions that remain largely unexplored in our research domain. However, some proposals have been made.

Borromeo Ferri and Blum (2010), for instance, suggested structuring teacher education around a four-dimensional model, one of which corresponds to the *diagnostic dimension*, focusing on the analysis of students' mistakes when working on modelling tasks (Blum, 2015, p. 89). For their part, Orey and Rosa (2018) analysed a pre-service teacher course on modelling where one of the categories included refers to what the authors call the "critical and reflective dimensions of modelling" (p. 177).

Blomhøj and Kjeldsen (2006) proposed the inclusion of an introduction to modelling and problem-oriented project work, followed by teachers' experiences with the implementation of modelling. This process is subsequently complemented by "reporting their reflections in a form that could be helpful for colleagues that would like to try out their courses or similar modelling projects" (p. 165). The authors identified several *dilemmas* for teachers in teaching modelling, such as the tensions surrounding its interpretation as an educational goal or as a means, or the challenge of fostering students' autonomy.

In all these cases, however, there is no systematicity in the characterisation of the constraints, they are explored and approached more as a personal issue, than a collective and institutional concern. In this paper, we start from the hypothesis that teacher education programmes should address the identification and analysis of *constraints* and *conditions* that may hinder or facilitate the teaching and learning of mathematical modelling. These conditions and constraints range from those that are specific to classrooms to generic ones referring to our society and school institutions. We will refer to this as the *ecology* or *ecological dimension* (as in Barquero et al., 2013).

Despite its importance, this dimension is rarely considered in mathematics education, even though implicit references can be found in research. Although the diagnostic dimensions are referred to in terms of "dilemmas", "obstacles", "beliefs", etc., this does not usually entail a substantial modification of the unit of analysis considered (Barquero et al., 2019), which mostly remains focused on the activities carried out within the classroom setting. Furthermore, there is no specific methodology that enables the analysis of the conditions and constraints for modelling, when considering

the *ecological dimension* in teacher education. This paper aims to be a step forward in this direction.

2.2 Theoretical considerations concerning the *ecology* of mathematical modelling

To analyse the set of constraints that hinder and the conditions that can facilitate modelling to be a self-sustained practice in mathematics education, we will rely on the *levels of didactic codeterminacy* (Chevallard, 2002), as presented in the framework of the anthropological theory of the didactic (ATD). In this paper, these levels (see Fig. 1) are used as as a common theoretical and methodological tool to categorise and locate the types of conditions and constraints identified by the participants into two different teacher education courses for modelling.

The scale hierarchy ranges from the most generic levels, those of *humanities*, *civilizations, societies*, and *schools*, to the most specific one, that of *disciplines*, in our case mathematics. The *specific levels* refer to the way a *discipline* is conceived and how it is structured into different *domains*, *sectors*, *themes* and *subjects*. The level of *pedagogies* refers to the conditions affecting the teaching and learning of different disciplines at the same time, that is, the way disciplines are organised to be taught at schools. The level of *schools* contains conditions and constraints concerning the status and functions assigned to educational institutions. The *societies* and *civilizations* levels concern the way societies understand the rationale and aims of knowledge and schools. Table 4 in Sect. 4.1 presents examples of conditions and constraints that can be located at the different levels of codeterminacy.

3 Research questions and initial hypotheses

This paper addresses an empirical study from a selection of two cases, corresponding to two in-service teacher education courses for the teaching of modelling at secondary school level. The two authors of this paper independently designed both courses in collaboration with other researchers-educators. The courses are based on different theoretical approaches to modelling, and different decisions were made regarding the tools to be transferred to teachers for the design and analysis of modelling. Despite the researchers-educators' initial differences, they all decided to include training activities centred on experiencing modelling (in which participants acted as "students"), designing modelling tasks (as "designers"), implementing them (as "teachers"), and conducting an a priori and *a posteriori* analysis (as "analysts"). This homogeneity allowed us to examine the decisions made by the participants when they worked on the design of modelling, as well as the conditions and constraints identified during the *a posteriori* analysis. The research questions addressed in this paper can be formulated in the following terms:

RQ1 How do the training strategies influence the decisions made and the conditions set out when the participants in each course work on the *design of the modelling tasks* and their a priori analysis?

RQ2 What *conditions* and *constraints* are identified by the participants in each course when they work on the *a*

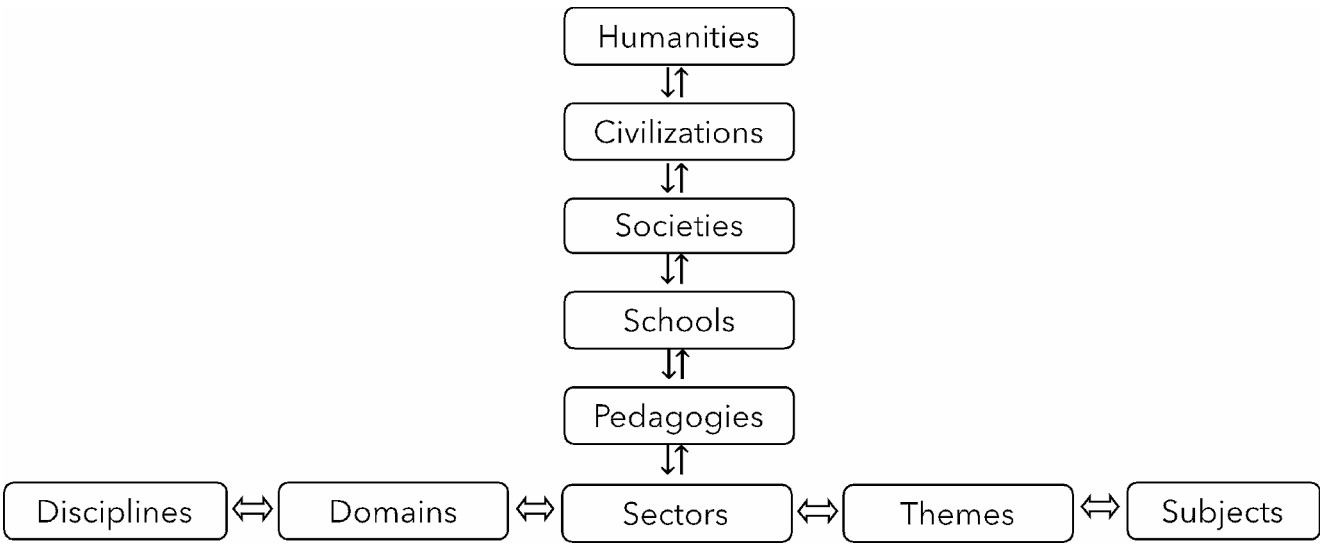


Fig. 1 Scale of levels of didactic codeterminacy. (adapted from Chevallard, 2002)

posteriori analysis, and at which level of the codeterminacy scale can they be located?

Given that both courses employ different training strategies and theoretical constructs to work on the design and analysis of modelling tasks, our hypothesis regarding RQ1 is that the training strategies and tools transferred influence the kinds of modelling tasks designed, as well as the levels of codeterminacy that are approached through these designs.

For RQ2, despite the characteristics of each course, we assume that both enable the teachers to identify significant and comparable conditions and constraints for modelling. Some of these conditions commonly emerge from the prevailing pedagogical paradigms in secondary education, while others are easily made visible as consequences of the instructional choices inherent to each course.

We start by presenting the characteristics of the two courses, followed by a synthesis of the conditions for their implementation (Sect. 4). Subsequently, we introduce the specific methodology employed to explore the participants' work in designing and analysing modelling tasks, along with the identification of conditions and constraints for the teaching and learning of modelling (Sect. 5).

4 Study context: characteristics of the courses and training strategies

To unify the description of both teacher education courses, we focus on describing their structure in modules, the time allocated, the nature of the activities, and the roles adopted by the participants (acting as students experiencing modelling, as designers, teachers, and analysts) and the educators (as lecturers or guides in the training). In both courses, the participants were in-service secondary school teachers from Spain (henceforth referred to as Course 1), and from Mexico and Colombia (Course 2).

4.1 Course 1: teacher education programme at the University of Valencia

Course 1 was conducted in Valencia (Spain), and was spread over three editions. The course consists of 30 h, with 10 h of direct training and 20 h of autonomous work. The course included 5 modules designed collaboratively by the second author and C. Gallart, an experienced mathematics teacher and modelling researcher.

Modules 1 to 4 A and 5 consist of 2-hour sessions of synchronous training. The initial four modules were conducted over four consecutive weeks. *Module* 4B constitutes a period of autonomous work, supported asynchronously by the educators, while the participants worked on the design of their modelling proposals. Table 1 provides an overview of Course 1.

Module 1 introduces the theoretical foundations of modelling, including modelling tasks, solution analysis, and competences associated with the modelling cycle. It draws on the description of the modelling cycle (Blum & Leiß, 2007), and illustrates the phases of the modelling cycle from the students' solutions. This module aims to show examples of modelling tasks. For instance, it provides the participants with tasks that encourage the solver to make decisions and to approach them from different strategies (for instance, the "driving school" task described in Achmetli et al., 2019). Fermi problems are also introduced, as an example of underdetermined tasks that require the introduction of realistic assumptions and the making of estimates (Ferrando & Albarracín, 2021).

Table 1 Structure of course 1: modules, phases, roles, and length

Focus	Role of participants	Role of educators	Time (h/min)
Module 1	Theoretical foundations of modelling		
Modelling tasks	Student/analyst	Guide	45 min
Perspectives on modelling for task design	Student	Lecturer	75 min
Module 2	Putting theory into practice: how to implement modelling?		
Modelling as a vehicle or as competence	Student	Lecturer	30 min
Description of a 5-lesson sequence for modelling	Designer	Lecturer	40 min
Balance between student independence and teacher support	Analyst	Guide	50 min
Module 3	Assessment of students' modelling performance		
Use of rubrics: a practical reflection	Analyst	Lecturer	60 min
Assessment instrument's design	Designer	Guide	60 min
Module 4 A	Conversations with actors		
Modelling from experienced students' perspective	Designer/analyst	Lecturer	120 min
Module 4B	Design, implementation and analysis		
Design and implementation of the modelling proposal	Designer	-	10 h
Reporting on the implementation	Analyst	-	10 h
Module 5	Pooling of experiences		
Open discussion	Teacher	Guide	120 min

Module 2 explores the practical aspects of implementing modelling tasks. Two complementary perspectives are introduced: modelling as a vehicle to introduce a particular mathematical content, and to develop modelling competence (Julie & Mudaly, 2007). Since educational programmes in Spain incorporate modelling as a specific competence, the module focuses on modelling as content. The educators explain how to design and implement sequences devoted to developing the modelling competence. The module addresses issues concerning the adaptive and strategic teacher intervention concepts (Stender & Kaiser, 2015) to enhance the balance between students' independence and teachers' support.

Module 3 covers modelling performance assessment tools, a concern for teachers (Blum, 2015). This module presents various assessment rubrics to evaluate each phase of the modelling process (Houston, 2007). At this point, the participants evaluate different students' solutions to a modelling task using various assessment instruments.

Module 4 A corresponds to a special session, as, in each edition, secondary school students and teachers with experience in modelling are invited. The exchange of ideas with the secondary school students is particularly interesting, as they provide a complementary point of view to that of the educators. Guest teachers share their experiences, particularly regarding task design. This session aims to help the participants define their interests to outline the design of the modelling activity to be implemented in their classes.

In *Module 4B*, the participants work autonomously to prepare the design of a modelling task, based on what they have learnt in the previous modules and, if possible, implement it. Each participant draws up a written report explaining the proposal. If the proposal has been implemented, the participants are asked to analyse the results, identifying conditions and constraints. Finally, *Module 5* focuses on the reflection of the participants' reports, under the guidance of the educators.

The main goal of Course 1 is to familiarise the participants with different modelling frameworks for designing modelling tasks and analysing students' work during their implementation in secondary school settings. Accordingly, each module introduces several modelling approaches to provide teachers with examples and tools to teachers for designing modelling tasks and evaluating students' solutions.

4.2 Course 2: teacher education programme in CICATA-IPN

This course is based on the five-module structure of the *study and research path for teacher education* (SRP-TE, Barquero et al., 2018). The course has run for several years since 2015/16, the experience in the academic year 2019/20 is here considered. This course was divided into two parts: the first part was developed in autumn 2019 and, the second in spring 2020. Each part spanned 5 weeks, amounting to 80 h of participant workload. The course ran exclusively online through Moodle, led by a team of 5 educators (2 members from CICATA-IPN and 3 from Spain), all researchers in mathematics education. The first author of this paper was part of the educators' team.

The participants worked in teams of 2–3 members and interacted through Moodle forums. Each team had a forum for each activity. Educator supervision consisted of presenting the activities through written statements and brief videos on the course website, offering feedback on the participants' discussions directly on the forums, and providing comments on the activities. The activities were structured as role-plays for participants: the role of student in *Module* 1, the role of analyst in *Module* 2, the role of designer in *Module* 3, and the role of teacher in *Module* 4. Table 2 summarises the organisation of Course 2.

Module 1 focuses on experiencing a modelling activity, as a student, based on the proposal of a study and research path (Bosch, 2018). The initial modelling questions were "*How are the sizes of T-shirts determined? What measurements are used in S, M, L, and XL sizes?*" These questions lead to looking for accessible data, organising and analysing data, constructing, and simulating numerical or geometrical models. In *Module* 1 A, each team worked on a report of their enquiry, sharing their results with another team. Then, in *Module* 1B, the teams were asked to include an analysis of the modelling process in their final report. At this stage, the educators introduced tools for the *epistemological analysis* of modelling, including specific terminology related to "systems" and "models" about the stages of the *modelling cycle* (Blum & Leiß, 2007; Borromeo Ferri, 2018; Chevallard, 1989; Serrano et al., 2009), and the use of *questions-answers maps* (QA-maps, Florensa et al., 2021) for the analysis of the modelling processes.

Module 2 focuses on the design of a *lesson plan* as close as possible to the teachers' practice, adapting the earlier T-shirt sizing modelling activity, to be implemented with their students. The participants initiated the process by elaborating individual lesson plans, to later discuss them with the rest of the team members to agree on a collective lesson plan.

In *Module 3*, the participants are asked to implement and observe their teaching proposal in a classroom setting. Since the second part of the course ran during the lockdown, the participants had to adapt the material to the new circumstances. This module ends when the participants can implement the lesson plan, at least its initial phases.

The SRP-TE finishes with *Module 4*, which involves the collective analysis of the implementations. The participants

Table 2 Structure of course 2: modules, phases, roles, and length

SRP-TE	Focus	Role of participants	Role of educators	# days / week(s)
Part 1 – Course 2 (5 weeks, 80 h)				
Module 1 A	Experiencing an SRP about T-shirt sizing			
	Experience of the SRP	Student	Guide	5d / w1
	Exchanging advances on the teams' reports	Student	Guide	3d /w2
	Final report	Student		2d / w2
Module 1B	Analysing the SRP			
	Tools for the analysis	Analyst	Lecturer	1d / w3
	Analysis of the experienced SRP	Analyst	Guide	4d / w3
Module 2	Adaptation of the SRP			
	Individual proposals of lesson plans	Designer		3d / w4
	Tools for the mathematical-didactic analysis	Designers	Lecturer	1d / w4
	Team proposal of the lesson plan	Designer	Guide	5d / w4-5
Part 2 – Course 2 (5 weeks, 80 h)				
Module 3 A	Design of material			
	Resources design according to the lesson plan	Designer	Guide	5d / w1
	A priori analysis of the resources	Designer	Guide	5d / w1
Module 3B	Implementation and in vivo analysis			
	Implementation and data collection	Teacher	-	5–8 d/w2
	Individual analysis	Analyst	-	5 d/w3
Module 4	*A posteriori* analysis of the SRP			
	Collective ecological analysis	Analyst	Guide	10d / w4-w5

were required to share a report on the implementations. They exchanged and compared these reports, along with the conditions created and the constraints found for modelling. The courses conclude with the final revised revision of the lesson plan by the teams, including a detailed *mathematical*, *didactic*, and *ecological* analysis of the modelling activity.

The main goal of Course 2 (following Barquero et al., 2018) is to support teachers to approach the institutional constraints affecting modelling-oriented teaching practices, while equipping them with new analytic and design tools to approach these constraints. The structure and activities of the SRP-TE's modules aim to make visible significant constraints for teachers.

5 Methodology and data analysis

The methodology followed in this study is predominantly qualitative. Our analysis was based on the reports elaborated by the participants from both courses. A total of 46 secondary school teachers (grades 7 to 12) completed *Course* 1 in one of the three editions. All participants attended all sessions and prepared a report with the design of a modelling task. Not all of them were able to implement their designs due to COVID restrictions. Finally, 20 participants were able to implement their designs and reports on their analysis.

A total of 19 participants took part in the first part of Course 2 and 12 in the second (some of them could not finish the course due to the outbreak of the pandemic). Three working teams (Teams 1, 2, and 3) were selected because of the richness of their work and reports. The data gathered correspond to the reports delivered by the participants in each course:

- Reports on the *design of the modelling tasks* to be implemented (in which the participants acted as designers). In Course 1, these reports (46 in total) were delivered individually at the end of Module 4B (see Table 1). In Course 2, the reports were prepared in groups (3 reports in total) at the end of Module 3 A (see Table 2).
- In Course 1, a total of 20 individual reports on the implementation of the modelling tasks were handed in (end of Module 4B).
- In Course 2, 8 individual reports including the analysis of the implementation were collected from the 3 selected teams (end of Module 3B).

5.1 Data analysis

Data collected from both courses are examined in accordance with the different levels on the scale of didactic codeterminacy (Fig. 1), which constitutes our main theoretical and methodological tool.

To address RQ1, we analyse the modelling tasks designed and the decisions made during the a priori analysis of these tasks. Our focus is on determining the level of specificity or generality at which the designed modelling tasks can be placed. For instance, those tasks involving only a specific mathematical content (e.g., the Pythagorean theorem) are at

Table 3 Categorical variables to analyse participants' excerpts

Participant code	The participants of Course 1 were coded with a four-digit format XX_YY, where XX is the year (19, 20 or 22), and YY is the participant number. The participants of Course 2 were coded with a four-digit format Tx_My, where x and y are the numbers of the team and of the team members.
Excerpt	Excerpt from the reports in the original language
Role	Role adopted by the participants: student, analyst, designer, teacher working in the classroom, teacher collaborating with other teachers
Nature condition/constraint	Condition as something modifiable by the teacher; constraints, on the contrary, as non-modifiable
Levels of didactic codeterminacy	Selection of 2 labels per excerpt made by the researchers. The 2 labels might correspond to 2 levels of the scale of didactic codeterminacy to which the condition or constraints better corresponds. Priority is given to the 1st and 2nd chosen labels.

the *subjects* level. Activities encompassing various contents, but all related to the same theme (e.g., measurement and estimation strategies) are placed at the *themes* level. Activities requiring the use of procedures from different themes (e.g., content related to the curriculum block on stochastics) are placed at the level of *sectors*. Tasks involving the connection of different sectors or domains within mathematics are placed at the *domains* or *disciplines* levels. Finally, tasks involving contents from other disciplines, other than mathematics, are categorised at the *pedagogies* level. The results of the analysis are presented in Sect. 7. Additionally, we investigate the conditions and constraints assumed by the teachers when acting as *designers* of these modelling tasks.

To address RQ2, the initial step is to identify excerpts in the selected participants' reports regarding the implementation of modelling in classrooms. For their analysis, we once again use the levels of didactic codeterminacy to locate the conditions and constraints identified by the participants. The researchers started by extracting excerpts from the selected reports. An excerpt was selected to report one condition or one constraint. This selection was initially made independently by each researcher. The selection of the excerpts was then exchanged to be validated by the other researcher. To facilitate this process and its analysis, the authors used a common Excel file with the same structure (see Table 3) to document the analysis.

The analysis, which followed a double-coding procedure, was initially conducted independently by each researcher and subsequently exchanged and validated. This approach facilitates the comparison and agreement of the researchers' decisions regarding the categorisation of the conditions and constraints. Several important aspects should be noted. Concerning the "role" and "nature", these variables were added after the initial independent analysis. Both researchers recognised that teachers tend to express themselves from different viewpoints. The participants often referred to their experiences as designers, as teachers guiding the implementation, or as teachers working with other teachers in the school/course, but also as analysts or students of the modelling activity (especially in course 2). Furthermore, the variable "nature" was also necessary, as the same participant could identify a constraint that hindered modelling and, a few lines later, express the same idea as a condition that they could create. This is consistent with the nature of conditions and constraints in the ATD, as a condition can become a constraint, and vice versa, depending on what is perceived as being able to modify. Regarding the levels of didactic codeterminacy, we began with an initial definition (as introduced in Sect. 2) that was later improved by sharing some examples of excerpts to refine the initial categories. Table 4 presents the initial definition showing some examples.

After the initial steps of the qualitative analysis, a total of 188 excerpts were extracted from the individual reports of a total of 28 participants who completed the implementation and its *a posteriori* analysis (96 excerpts from Course 1, 92 from Course 2). To facilitate a quantitative analysis, the categorical variables of "role", "nature", and "levels of didactic codeterminacy" were coded. Coding them has enabled us to perform an analysis focused on comparing the relative frequencies of the values of each variable in the two courses. The aim was twofold: first, to provide a classification of the conditions and constraints identified by the participants in each course, and second, to identify emerging themes within each level of the scale of codeterminacy. Both aspects are discussed in the following sections.

6 Results concerning the decisions made and conditions set out on the design of modelling tasks

6.1 Course 1: variety of modelling tasks at different levels of specificity

When considering the level of codeterminacy that corresponds to the modelling tasks proposed by the teachers, we found that of the 46 participants, 11% proposed tasks that focus on *specific content* (*subject* level). These tasks are relatively limited and involve a direct *application* of a mathematical concept. For instance, participant 22_07 proposed the "*Shoelaces task"* about comparing three ways of tying shoelaces to find the most efficient one. The task included simplified images of shoelaces, with the mathematical interpretation of the real context. In his report, the participant

Table 4 Initial definition of the categories and examples

Levels of codeterminacy	Initial definition "Conditions and constraints for modelling…" and examples
Societies	Concerning how societies understand the rationale and aims of school institutions, and the specific role of mathematics in society. *One difficulty lay in determining which sources of information were reliable, as there were many blogs and videos, but neither the students, nor I were sure about their veracity* [C2, T1_M1, constraint].
Schools	Emerging from school as a social institution, which particularly affects what the teacher and the students can do at school and in the classroom. *The students could have used computers for data searching. However, they did not consider this possibility, as if it was forbidden in the school* [C1, 22_05, constraint]
Pedagogies	Affecting the teaching and learning of the different disciplines, that is, the way disciplines are defined, connected, and diffused in schools. *The presentation of a modelling task like the one designed based on everyday life and outside the scope of mathematics, that encourages group work is an appropriate way to involve students in their learning* [C1, 22_09, condition]
Disciplines (and the specific levels)	Referring to how the discipline (mathematics, in our case) is organised or structured into subdivisions (domains, sectors, themes, subjects) in an educational institution. *Exploring modelling questions facilitates connecting knowledge of several themes in mathematics at the same time and for the same purpose* [C2, T3_M2, condition]

stressed a 4-phase approach to solving the task: understanding the situation, identifying mathematical strategies, using mathematical procedures, interpreting and validating solutions. This activity aligns with the *educational perspective* (Kaiser & Sriraman, 2006) and represents a simplified version of a task addressing specific mathematical content.

Other participants designed modelling tasks that involved different content, and that can be placed at a more general level related to mathematics *themes* or *sectors*. Approximately 17% of the participants proposed a sequence of tasks focusing on measurement and estimation. These types of Fermi problems (Segura & Ferrando, 2023) were explicitly introduced in the course. The *models and modelling perspective*, with the notion of model development sequences (Ärlebäck & Doerr, 2018) was also introduced. Participant 19_04 designed and implemented a "*Fermi sequence*" consisting of five real-context tasks involving estimation and measurement procedures. The first one corresponds to the straw bale task (Borromeo Ferri, 2018, p. 14). The second and third were problems about estimating the number of elements in a bounded surface and volume. The last two were real-context estimation tasks, but in that case, the models were more complex.

One aspect highlighted in Course 1 was that modelling should promote the integration of different mathematical contents, even from different themes, emphasising the role of modelling in accessing the level of *sectors* and *domains.* Approximately 24% proposed tasks involving concepts from different topics within a mathematical *sector*. For instance, one participant designed the "*Gold task*", in which students were asked to reflect on it is possible to predict the rise in the price of gold to decide whether it is advisable to invest or not. This task involved statistical concepts. This task was aligned with the *realistic perspective* (Kaiser & Sriraman, 2006).

When analysing the idea of trying to integrate different mathematical contents in a broader sense (then trying to work on the *domain level*), we found that 20% of the participants aimed to integrate mathematics in a wide manner. For instance, as explained by participant 19_08:

> I aimed to help students develop competence in solving real-life problems without considering what mathematical content the students were going to learn. [C1, 19_08]

This participant designed the "*irrigation system task*" whose starting point was the real problem of renewing the irrigation system for a playground. The students were asked to design an optimal irrigation system (in terms of cost and water consumption).

We found a high ratio of participants (28%) that designed tasks that were outside the scope of the mathematics discipline. For instance, one participant designed the "*volcano sequence*", which included tasks related to a genuine relevant situation: the eruption of the volcano on the island of La Palma. As he explains:

> You should use the knowledge of mathematics and biology that you think is appropriate, as you have all seen it in class. [C1 22_22]

In the task, students were asked about the estimated cost of distributing basic food rations to all the inhabitants that were evacuated from their homes. This proposal is in line with a *socio-critical perspective* (Barbosa, 2006).

In summary, and concerning RQ1, Course 1 adopts a more prescriptive approach, introducing a diversity of modelling tasks based on various modelling perspectives. This course can be characterised as a *push training approach*

(Hargreaves, 2013), where the primary strategy is to equip participants with diverse tools and approaches that they then individually apply in designing modelling tasks. This decision affects the variety of tasks designed, which can vary from those focused at the most specific levels (those of *subjects* or *themes*) to those more generic, beyond the mathematics *discipline*. Despite the wide diversity of tasks, and the range of levels of coterminacy they interpellated, an important limitation arose when attempting to share and compare the designs that are supported by different modelling approaches.

6.2 Course 2: work on the adaptation of the same initial question of the SRP

In Course 2, the instructional strategy adopted was different. In Modules 2 (as *analysts*) and 3 (as *designers*), the participants were asked to analyse and work on the redesign or adaptation of the same generating question "*How are the sizes of T-shirts determined?"* They had experienced this as students. All the teams worked on the same modelling task, which can be placed at a generic level outside the scope of the mathematics discipline, about T-shirt sizing, involving the interaction of several disciplines (statistics, mathematics, anthropometrics, business administration, fashion design, etc.).

As for the *analysis* of the modelling task, after experiencing it as students, the educators asked the participants to sketch the modelling path followed by using the *questions-answers map* (QA-map). This way of describing the modelling process provided the participants with a description of the modelling process, and it appeared as a tool to sketch out the disciplines, domains, and themes that could appear during the SRP. The following questions are examples of the ones described by *Team* 1 when analysing their experience.

> Q_1: What T-shirt sizes does each brand produce? Are there differences between the different brands? Are there standard measurements for each size?
>
> $Q_{1.1}$: What mathematical models do different brands follow to decide on the sizes? $Q_{1.2}$: Do all brands follow the same models?
>
> Q_2: What variables are necessary to determine a T-shirt size? How are they related?
>
> $Q_{2.1}$: Do all brands consider the same variables and measurements? $Q_{2.2}$: What relation exists (if any) between the different measures?
>
> Q_3: How do size patterns vary over the years?
>
> $Q_{3.1}$: Do anthropometric factors change over time?
>
> $Q_{3.2}$: What factors influence the evolution of fashion?

When working on the design of the SRP, the teams started with the same initial question about T-shirt sizing. The participants were explicitly introduced to the SRP proposals and to a common methodology and tools for their design. The educators presented some papers about the traits of the SRP in a change of paradigm for the teaching and learning of mathematics and some tools for their design (such as Chevallard, 2015 and Serrano et al., 2009). The participants were asked to use some of those tools to work on the design of the adapted version of this SRP.

The teams began working on the mathematical design of the SRP, adapting the QA-map. They tended to assign names to the different stages of the SRP and of the modelling process to help the students understand at which step they were in the activity. Comparing their proposal to their previous experience, the teachers tended to limit the SRP to curricular-recognised knowledge, replacing specific modelling terms (systems, models, hypotheses, patterns, testing, etc.) with curricular-related ones. Representative questions from the proposal of the QA-map prepared by *Team* 1 are shown below.

> Q_0: Have you ever wondered how brands establish the measurements of T-shirt sizes?
>
> Q_1: What sizes do the different brands work with? What differences are there? [*Seeking expert opinion*| *Search for data*]
>
> Q_2: Do all brands use the same parts of the body to indicate T-shirt sizes? What kind of data are comparable? What data do we select to model what different brands do? [*Analysis of the information*| *Measurements comparison*| *Use of statistical tools*]
>
> Q_3: What conclusions can be drawn from the statistical analysis? Can we say that there are proportionality relationships? What does the comparison of medians, means, and deviations tell us? [*Interpretation of statistical analysis*]
>
> Q_4: What would be the ideal sizes? How do a country's anthropometric measurements influence this? [*Preparing the final response*]

The participants, when working as designers, often reproduced what they were used to doing: linking activities to curriculum labels. This may be interpreted as an important constraint imposed by school institutions: curricula described in terms of notions and procedures that should be covered in a short period. One member of *Team* 1 explicitly expressed this constraint:

> The numerical data should be studied using tools of descriptive statistics. Mathematical tools, such as class intervals, continuous variables, arithmetic mean,

Table 5 Role assumed by the participants when describing the conditions and constraints encountered

	Student	Designer	Teacher implementing	Teacher sharing experiences
Course 1	0%	16%	76%	8%
Course 2	14%	4%	70%	12%

> variance, standard deviation, least-squares adjustment, linear mathematical models, scatter plots, correlation, etc. should be used. These are important concepts to be explained in our SRP [C2, T1_M1].

To sum up, and concerning RQ1, the strategy followed in Course 2 is clearly different to Course 1. On the one hand, one main theoretical approach is here considered, that of the SRP as proposed in the framework of the ATD. As explained by Barquero et al. (2018), the Course 2 option consisted of preparing, together with the in-service teachers, a shared mathematical and didactic reality to be used as a confrontation device for the teaching proposals and the theoretical tools introduced in the course. Course 2 can be more closely associated with a *pull training approach* (Hargreaves, 2013), where participants progress throughout the training, while they are provided with new tools for the design and analysis of modelling. On the other hand, all participants, within their working team, work on the same SRP about the T-shirt sizing. This can be initially interpreted as a limitation, but this decision also facilitated that participants could share and compare their proposals, rather than diversifying into different modelling tasks.

7 Results about the conditions and constraints identified in the *a posteriori* analysis

7.1 Roles assumed by the participants

The analysis of the excerpts from participants' reports reveals that the course participants assumed different roles when describing the conditions and constraints for modelling.

In Course 1, most comments (76%) are made assuming the role of the *teacher* implementing the activity, a significant proportion of the conditions and constraints (16%) are identified from the role of the *designer* (see Table 5). This reflects that the participants were free in the design of the tasks, which caused a certain degree of uncertainty. It is worth noting that there are no excerpts from the student's role, possibly because in the design of the course the focus was not so much on solving modelling tasks, but on introducing and reflecting on the types of tasks.

In Course 2, most of the comments (70%) are made when adopting the role of teachers implementing modelling. There is also a considerable percentage of conditions and constraints identified under the role of teacher collaborating with other teachers (12%), or assuming the role of student experiencing modelling (14%). Only 4% of the excerpts are described from the designer's perspective. As mentioned earlier, this course places a strong emphasis on experiencing a modelling activity as students, based on an SRP, to then move together to work on the analysis and to work on its redesign. This may explain why the participants in Course 2 identify conditions and constraints from the role of the students to a greater extent than the participants in Course 1.

7.2 Reflections on the identified conditions and constraints

Concerning the nature of excerpts as conditions or constraints, about half (54%) of the excerpts in Course 1 are related to conditions the participants considered they were able to create for modelling. The rest (46%) of the excerpts were formulated as constraints encountered. Table 6 illustrates the relative frequency of excerpts referring to conditions or constraints located at the different levels of codeterminacy.

More than 80% of the conditions and constraints are placed at the levels of *disciplines* and *pedagogies*, whether as first option (Label 1) or second (Label 2). The rest are distributed among the upper levels of *societies* and *schools*, or in the lower or *specific levels* of school mathematics. In the latter case, the participants who focused more on the *specific levels* were mostly those who tended to design modelling tasks that were close to these specific levels. This raised concerns about the rigidity of topics in curricula, and about the difficulties students have with some concepts. For instance, a participant explains:

> The students have encountered difficulties, as they do not understand basic concepts of geometry (surface calculations) [C1, 20_03].

With respect to Course 2, Table 7 indicates the relative frequency of excerpts referring to conditions or constraints located at the different levels of codeterminacy.

Regarding the nature of the conditions or constraints identified by the participants, the situation is similar to Course 1. About 70% of the conditions and constraints

Table 6 Relative frequency of the selected excerpts in course 1

	Societies	Schools	Pedagogies	Disciplines	Specific levels
Label 1	3%	6%	**43%**	**46%**	2%
Label 2	1%	7%	**42%**	**39%**	11%

Table 7 Relative frequency of the selected excerpts in course 2

	Societies	Schools	Pedagogies	Disciplines	Specific levels
Label 1	**6%**	**24%**	**28%**	**41%**	1%
Label 2	8%	8%	**45%**	**27%**	12%

identified by the participants in Course 2 are placed at the levels of *disciplines* and *pedagogies*, lower than the more than 80% in Course 1.

However, a significant percentage refer to a more generic level. A total of 30% can be located at the levels of the *school* (24%) and *society* (6%), in comparison with less than 10% at these levels in Course 1. This can be explained by the fact that participants frequently discussed a shift in focus towards the inquiry into questions, consistent with the educators' training intentions with the use of the SRP within the paradigm shift for modelling. As explained by T2_M2 referring to curricula constraints:

> An important constraint is the curriculum, as we [teachers] are governed by a rigid, strictly time-guided syllabus that gives small room to inquiry-based teaching with our students as an official part of mathematics subject. [C2, T2_M2]

When the participants adopt the role of designers or teachers implementing modelling, some of the initially identified conditions for modelling (such as addressing open authentic questions) turn into constraints for their designs. Following with T2_M2:

> Inquiring into questions rather than introducing mathematical content is totally out of the ordinary teaching. Students are used to being introduced (by the teacher) to concept after concept […]. But, what happens now is that, instead of concepts and a list of tasks to practice, students have to address open questions. They crash against their understanding of what mathematics is supposed to be.

8 Discussion and conclusions

The *ecological dimension* is often overlooked in mathematics education. While some references are made to it in research on mathematical modelling, for instance concerning the diagnosis dimension (Borromeo Ferri, 2018), specific tools are rarely proposed. This paper has aimed to contribute to the understanding of teacher education for modelling by examining the course designs, their training strategies, the kinds of tools transferred to in-service teachers, and the significance of addressing the *ecology* of mathematical modelling. By considering two teacher education courses for modelling, developed in different contexts and supported by different theoretical approaches, we have explored the commonalities and differences of the instructional design and strategies to broaden teachers' knowledge of modelling and its ecological questioning.

8.1 About the design of the modelling tasks and their a priori analysis

If we shift our attention to the decisions made and conditions established when the participants engaged in designing modelling tasks in the two courses, some notable differences emerge.

Course 1 approach is focused on empowering the participants to design, implement, and analyse modelling proposals guided by different theoretical frameworks. The participants were instructed to make decisions on aspects such as the selection of modelling tasks, determining which theoretical framework facilitated their designs. Based on the results, it can be inferred that the participants considered a wide variety of modelling approaches with consequences for the type of tasks designed. However, the variety of individual designs produced in this course did not facilitate an in-depth discussion and comparison of the *ecological* conditions for modelling and their dependence on the teaching institution.

Course 2 is based on the proposal of the SRP for teacher education. The proposal of a module structure using role-play, from students experiencing an SRP to teachers adapting an SRP, appeared to be a successful strategy to create a rich *milieu* (Brousseau, 1997) between the participants and the educators. The course combines individual and group work to facilitate the emergence of new needs among the participants. Epistemological and didactic tools for designing and analysing modelling and its ecology are progressively introduced in the training process. However, this approach also has limitations, as the participants gain experience in the design and adaptation of the same (or similar) SRP based on the same modelling task.

8.2 About the conditions and constraints identified in the a posteriori analysis

Despite the differences between the two courses, the participants identified several common conditions and constraints for modelling. As indicated by the results, presented in Sect. 7, the two codeterminacy levels that obtain the highest number of conditions or constraints are the levels of *disciplines* and *pedagogies* (with 85% in Course 1 and 70% in Course 2).

Concerning these levels, it is important to stress the common themes. Focusing on the level of the *disciplines*, the most frequently mentioned constraints refer to the necessary changes in the mathematical responsibilities assumed by teachers and students, that is, changes in the *didactic contract* (Brousseau, 1997). Indeed, a major difficulty for teachers is the initial confusion of students due to their unfamiliarity with modelling.

> Learners may feel they are wasting time because, in this problem, there is no quick response like the ones they are used to address. [C1, 20_11]

However, this initial confusion helps the teachers reflect on how working with modelling affects students' roles, and consequently, teachers' roles in more student-centred environments (Blum & Leiß, 2007).

> Monitoring the activity is no longer entirely in the hands of the teacher, who is no longer at the centre of the didactic situation […]. The teacher should adopt a guiding profile that allows her to hand over the leadership of the class to the students. [C2, T3_M2]

Noteworthy, some comments highlighted the necessity of creating *new terminology to talk about modelling*. This need became apparent when participants acted as students experiencing modelling, or when students in classrooms were asked to report on their modelling process.

> It is common for students not to be used to expressing themselves mathematically. They find it difficult to use mathematical language to explain phenomena in their daily lives. It is the teacher's job to build this confidence by guiding them in this process. [C2, T1_M1]

Concerning the *pedagogical* level, one of the most recurrently identified constraints is related to *time management*. As expressed by one participant:

> I think students need more time. They need to have a work plan that shows what needs to be done before they start looking for information. The students tended to start working very quickly without having a clear idea about what to do. [C1, 19_02]

Furthermore, the participants in both courses highlighted the difficulties in organising and managing teamwork, emphasising the importance of sharing and comparing the different teams' progress.

> The students came up with different ways of achieving the result. Sharing and contrasting opinions also helped the students validate their solutions. [C1, 22_05]

At the *pedagogical-school* level, another constraint concerns the need of *breaking the boundaries between school disciplines*, typically established by curricula. As expressed by one participant:

> Another important point is the existing curriculum, as we are regularly governed by a rigid and tight syllabus that hardly allows us to carry out real research with our students. [C2, T2_M2]

When comparing in detail the results from both courses (Tables 5 and 6), one might note also relevant differences. On the one hand, the conditions and constraints placed at the *school* levels vary significantly in Course 2 (from 6% in Course 1 to 24%, Label 1). Moreover, the conditions and constraints detected at the *society* level are also more frequent in Course 2 (3% in Course 1 vs. 6% in Course 2, Label 1), while participants in Course 1 seem to remain at the level of the *discipline* and the *organisation of contents* (39% vs. 27%),

This can be explained as Course 2 presents a modelling task, based on an SRP, that strongly modifies the prevailing pedagogical paradigm and didactic contract for the teaching of mathematics, compared to Course 1. Many of the discussed constraints can be related to the dominant paradigm of "visiting works" (Chevallard, 2015). In this paradigm, mathematical activity is primarily associated with the visit of existing works (mathematical objects, tools, or concepts) giving a secondary role to modelling and inquiry in mathematics teaching.

Our research emphasised the need to identify and study constraints, as a crucial step to overcome them, and to ensure modelling as a school sustainable activity. Teacher education must facilitate collective awareness about the existence and scope of the institutional constraints limiting modelling practices. To work in this direction, the results from the two courses show the need to introduce more specific tools, and better-controlled training strategies to properly identify and discuss the kinds and the scope of constraints and conditions observed. Our results are in line with the conclusion from the international study conducted by Dorier and García (2013). In this regard, this paper provides insights into the future design of teacher education actions. Specific training strategies, such as comparing the *a posteriori* analysis of different implementations, giving "names" to the conditions/constraints identified, or introducing specific instruments, as

the levels of didactic codeterminacy, could help the design of future teacher education initiatives.

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Declarations

Compliance with ethical standards The authors declare there are no potential conflicts of interest.

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