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Devising a plan to solve Fermi problems involving large numbers

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Abstract Fermi problems are open, non-standard problems that require the students to make assumptions about the problem situation and to estimate relevant quantities before engaging in, often, simple calculations (Ärlebäck *The Montana Mathematics Enthusiast*, 6(3):331–364, 2009). Our study focuses on the plans—schemes aimed at solving—devised by students aged 12–16 years to solve Fermi problems involving big numbers. The strategies which appear in these schemes are characterized herein and analyzed to find out to what extent they would be suitable to solve the problems. Some of the identified strategies are related to magnitude estimation (use of a reference point), whereas others correspond to modeling processes (grid distribution, use of a concentration measurement, reductions of a problem, and use of a proportion). We conclude that Fermi problems involving large numbers may constitute a good opportunity to discuss heuristics and problem solving strategies in compulsory secondary education classes.

Keywords Mathematical modeling · Compulsory secondary education · Estimation · Problem solving

1 Introduction

In recent decades, a large amount of studies have been carried out in order to understand the processes by which students learn to solve problems. A wide review of the advances made in this field can be found, for instance, in Lester (1994), Lesh and Zawojewski (2007), or Schoenfeld (2007). However, as pointed out by Arcavi and Friedlander (2007), even though a large number of elements and factors involved in problem solving have been identified, a large part of the attained theoretical knowledge has not yet been transferred to the classrooms.

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With the aim to contribute with ideas related to problem solving in class that may be useful to mathematics educators, our work is focussed on the plans devised by a group of students aged 12–16 years to solve Fermi problems which require estimating large amounts.[1] We have specifically studied the stage which Pólya (1945) designates as *devising a plan*. Most studies conducted on problem solving focus their attention either on the final stages of the process or on resolution as a whole. The stage in which the problem solver devises a plan has been little studied, and we therefore intend to address this part of the process in order to contribute with new ideas to link theory with the classroom.

The problems we presented to the students are Fermi problems (Ärlebäck, 2009) involving large numbers. Since they are problems the students seldom come across in class, solving them requires them to get involved and devise their own strategies, while it also requires them to include these strategies in their mathematical knowledge base. According to Schoenfeld (1992), this can play an important part in making students become competent problem solvers.

2 Theoretical framework

We herein introduce specific aspects of the existing literature that are relevant to our study in relation to problem solving, estimation, and Fermi problems.

2.1 About problem solving

In his seminal work, Pólya (1945), based on introspection, examined the behavior of an ideal problem solver, establishing that he would linearly go through four stages—(a) *understanding the problem*, (b) *devising a plan*, (c) *carrying out the plan*, and (d) *looking back*—moving on to the next stage only when the previous one is complete. In this study, we focus on the second stage of the problem solving process by collecting information on the schemes aimed at solving submitted by the students for a specific kind of problems.

According to Pólya, "the problem should be well chosen, not too difficult and not too easy, natural and interesting," (Pólya, 1945, p. 6) therefore choosing the problem accordingly is an essential part of the project. It is also important to make sure that the student can understand the verbal formulation of the problem and point out its most relevant aspects. The student should then focus on the main parts of the formulation and start on the first procedures designed to obtain a solution. In the next stage, devising a plan, the problem solver should elaborate an action plan, which may be an idea which forms gradually or that develops after several failed attempts to come up with a scheme. Since ideas do not usually form without any previous knowledge on the subject, Pólya suggests asking the following question to the student "Do you know any related problems?" (Pólya, 1945, p. 9) so that students can establish relationships with their existing knowledge and experiment with other similar problems. When the student manages to elaborate a plan to solve the problem, he will be ready to face the two last stages of the process: *carrying out the plan* and *looking back*.

Other authors have introduced other metacognitive models for the solving process of a problem. According to Verschaffel, Greer, and De Corte (2000), these models follow the structure presented by Pólya (1945) and include several stages, which do not always have to be followed linearly: creating a mental image of the problem, planning how to solve the problem, interpreting the obtained data and developing an answer, and evaluating the solution. Carlson

[1] This paper is an extended and elaborated version of a research report by Albarracín and Gorgorió (2012) presented to PME36.

and Bloom (2005) state that problem solving has a cyclic nature. Their study carried out on professional mathematicians has shown that the resolution process can be divided into four stages, equivalent to those of Pólya: *orienting*, *planning*, *executing*, and *checking*. These authors observed that in the initial phase, *orienting* stage, the mathematicians went to great lengths to ascribe meaning to the information provided, and their approach was conscientious, constructing logical representations using diagrams, graphs, or tables and attempting to classify the problem by placing it within the context of other problems they already know. They also observed that, in the second stage, *planning*, mathematicians used their heuristic abilities and engaged in self talk to verbalize their speculations, while trying to evaluate their potential based on previously acquired knowledge.

At the beginning of the 1980s, a new trend appeared, encouraged by the National Council of Teachers of Mathematics (NCTM). It focussed mathematics lessons on problem solving from the specific study of different heuristics. However, Schoenfeld (1992) admits that the study of heuristics is not enough for the students to become competent at problem solving and claims that other factors intervene in this process. Such factors are, among others, metacognition and various emotional aspects or the student's belief system. In addition, Schoenfeld explains that a reason for this lack of success could be due to the fact that students do not relate to the studied heuristics and therefore do not use them in their solving process. Schoenfeld suggests helping them to develop a large number of more specific solving strategies and to establish clear links between different kinds of problems. From this perspective, in our work, we would like to identify what type of strategies the students are able to develop by themselves, so that the knowledge we may gain can be used by teachers as a starting point when working on problem solving in class.

2.2 About estimation

By estimation, we mean a rough calculation or judgment of the value, number, quantity, or extent of something. The noun has a Latin root, *aestimare*, and originally, it meant "intellectual ability, comprehension" linked to the idea of "valuing." Therefore, the value which results from the estimation will depend on the person who makes the judgment or calculation.

Going over the literature may lead to finding references to different types of estimation, since there is a great variety of tasks referring to the concept of estimation which come from various sources and do not share the numerical patterns which enable their execution (Booth & Siegler, 2006). Particularly, Hogan and Brezinski (2003) distinguish between numerosity, measurement estimation, and computational estimation.

Numerosity refers to the ability to visually estimate the number of objects arranged on a plane; estimating measurements is based on the perceptive ability to estimate lengths, surface areas, time, weights, or other similar measurements of ordinary objects, and computational estimation refers to the process by which the value of a calculation is approximated. Certain studies address the estimation strategies that apply to each of these meanings.

The study of the estimation of numerosities reveals strategies that often involve a mix of counting and sensing quantity without counting (see, for instance, Sophian & Chu, 2008). Other studies, such as Ginsburg (1991) or Barth, Kanswisher, and Spelke (2003), which are focussed on the perceptive and visual components of estimation and numerical representation, describe some strategies that can be used in other kinds of estimation, such as calculating half the objects in a set and multiplying them by two or iterating a unit as a reference point.

A review of the existing literature shows that students find computational estimation difficult because they are not aware of the numerical meaning and the processes involved in estimation (Hanson & Hogan, 2000; Lemaire & Lecacheur, 2002; Verschaffel, Greer & De Corte, 2007).

Some of these processes are methods which correspond to computational estimation such as truncation, rounding, or numerical compensation (Reys, Rybolt, Bestgen & Wyatt, 1982). Amongst the existing research on estimation of continuous magnitudes, it is worth noting Hildreth (1983), Jones, Taylor, and Broadwell (2009), and Jones, Gardner, Taylor, Forrester, and Andre (2012), which study the strategies and processes that estimators use when facing different situations.

Hogan and Brezinski (2003) relate numerosity to the estimation of magnitudes and claim that they make up one only kind of estimative ability. According to these authors, both types of estimation share a strong spatial component and both depend on several perceptive abilities. Hence, numerosity and magnitude estimation differ from computational estimation that depends on exact calculation abilities and numerical judgment. Regarding magnitude estimation and numerosity, Hogan and Brezinski (2003) claim that "competence in measurement estimation does not follow automatically from general mathematical development. Measurement estimation should be addressed as a separate skill" (p. 276).

So far, we have introduced three kinds of activities, included within the concept of estimation, which have their own characteristics and relations as described in the literature. However, there is yet another type of estimation activity which has not been dealt with explicitly in the literature. This type consists of the calculation of values in predictive activities or the estimation of a reality by the use of models which represent the actual situation.

Lesh and Harel (2003) define the concept of model as follows:

> *Models* are conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors. Their purposes are to construct, describe or explain other system(s).
> Models include both: (a) *a conceptual system* for describing or explaining the relevant mathematical objects, relations, actions, patterns, and regularities that are attributed to the problem-solving situation; and (b) *accompanying procedures* for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals (p. 150).

In the absence of a specific procedure to exactly determine the value of an unknown magnitude, the situation may be reduced to a simplified one that represents it. This would be the case of the estimation of the amount of soil to be moved in the construction of a stretch of motorway or the case of predictions such as the growth of the gross domestic product of a country. There are also real-life situations which do not correspond to a well-defined and unique value but possess a range of acceptable values due to their complexity. Hence, the time required to get to the bus stop depends on the walking speed or on external factors such as the pedestrian traffic light being green when one comes to it. The representation made when simplifying a situation corresponds to a mathematical model which does not allow for an exact answer but an approximation, the validity of which depends on how much the chosen model corresponds to reality. In this sense, the obtained result is an estimate, since it requires a judgment of value based on the results obtained from the model.

In our study, we use problems related to this kind of estimation activity that links magnitude estimation to the representation of real life through a model.

2.3 About Fermi problems

Ärlebäck (2009) defines Fermi problems as "Open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities

before engaging in, often, simple calculations" (p. 331). This definition does not fully describe the entirety of Fermi problems. However, it includes some of the main ideas which characterize this type of problem, such as the possibility to be broken up into smaller problems to be solved separately and by means of estimation.

Fermi problems are characterized by not presenting all the information which would apparently be necessary for solving them and by being difficult to solve by using specific methods. The classical example of a Fermi problem is the estimation of the number of piano tuners in the city of Chicago. On the basis of the estimation of the total population of Chicago, the proportion of families in Chicago that may have a piano, and the time needed to tune a piano, the solution may be reconstructed by means of conversion factors (see Efthimiou & Llewellyn, 2007, for more details). This process is based on designing a problem solving strategy adapted to the problem which consists of breaking it into subproblems that may be solved separately by means of their respective estimations. In order to solve a Fermi problem, a simplification process of the formulated situation is required.

From the perspective of their importance in the teaching of mathematics, it is not enough to consider that Fermi problems are related to estimation tasks. An additional argument for using Fermi problems in the classroom is the possibility to use them as a bridge between mathematics and other school subjects, allowing students to approach different interdisciplinary tasks. According to Ross and Ross (1986), another reason for introducing Fermi problems in class is to offer the students the view that "doing mathematics is not always about getting exact answers through well-defined procedures" (p. 175). Peter-Koop (2004) claims that Fermi problems are better and more useful when not treated as purely intellectual exercises and are focused on real-life situations, within an everyday-life context. In her work with primary school students and Fermi problems, Peter-Koop has proved that different students solve problems in very different ways and that they generate new mathematical knowledge in the problem solving process. Ärlebäck (2009), in his study of the processes students use to solve Fermi problems, observes that they promote the students' discussion on the mathematical component of situations related to everyday life and that they may be presented individually or in groups, and at various school levels, since these problems do not require any specific mathematical knowledge. However, the way the activity is set out should be planned carefully, as Chapman (2006) points out by observing that, in most cases, teachers present these problems in a rather restrictive manner which does not allow for a narrative analysis of the given situations.

3 The study

Our study focuses its attention on the plans devised by students aged 12–16 years when facing Fermi problems involving big quantities. We have therefore chosen a set of problems and asked a group of students in compulsory secondary education to elaborate schemes to solve the problems which will be analyzed from a qualitative point of view. By scheme, we mean an outline of a plan to solve the problem, as opposed to a summary of the process followed to solve it. The goals of the study are as follows:

1. To characterize the strategies which appear in the schemes presented by the students to solve the problems, and
2. To determine the extent to which these schemes would be appropriate to solve the problems posed.

3.1 The problems

In this study, we have used Fermi problems, formulated in contexts the students are familiar with, which demand estimating large quantities. This type of problem is not being considered in the current Spanish curriculum, and thus students are not being taught particular methods by which to solve them. This is why the stage of planning becomes important in the solving process. For the same reason, we understand that the study of the schemes devised to solve the problems may provide useful ideas to the teachers for them to design their teaching activities.

By definition, Fermi problems can be restructured by breaking them up into smaller problems that are easier for the students to approach than the original problem, a characteristic that connects with one of Pólya's recommendations. This process in itself is part of the elaboration of a scheme to solve the problem, and we therefore consider Fermi problems suitable for our study. Moreover, along the lines of Van Den Heuvel-Panhuizen (2005), we hope that the realistic context introduced in the formulations will prompt different types of strategies in the students. Dealing with large quantities does not allow for simplistic approaches to the solution, and therefore, we would expect students to devise strategies including a great deal of mathematical elements from the abstract representation of the reality studied.

This study starts from an initial list of 36 problems formulated in different contexts which pose questions related to different kinds of continuous and discrete magnitudes. Some of the problems are original, while others are variations or adaptations of others collected from several lists[2] that make up an initial choice of rather heterogeneous problems. From this first list, we decided to focus on problems that dealt with discrete magnitudes that involved counting the number of objects or people. These problems were chosen for being more within the students' reach than those that involve continuous magnitudes.

We are interested in problems which have an unambiguous wording for the students and can be approached from different angles. Based on an evaluation performed by experts, we selected 16 of the 36 problems to be used in a pilot test. A worksheet was prepared for each of them, where the problem was contextualized and for which the students were asked to provide a solution action plan in writing. Each of the worksheets was handed out to six students from different levels of compulsory secondary education, and their answers were collected and analyzed.

This pilot test enabled us to gather information to detect which problems provide better results, with a greater wealth of information. We also observed that when formulations require the students to give an estimate for the requested amount, they focus on reaching a solution and forget about writing the plan devised to solve the problem. For this reason, we decided the final questionnaires should explicitly ask for just one description of the plan in writing, since this study focuses exclusively on the stage *devising a plan* of the solving process.

The following five problems were selected after the pilot test.

- *Problem A:* If a concert is being held at the high school, how many people can the high school playground hold?

[2] The initial list of problems is derived from the following references:

- *A Collection of Estimation Problems* in *Fermi Problems Site* of the University of Maryland. In http://www.physics.umd.edu/perg/fermi/fermi.htm, last visited on 04 April 2012.
- *Classic Fermi Questions with annotated solutions* available online at http://iws.ccccd.edu/mbrooks/demos/fermi_questions.htm, the website of the *Collin County Community College District*, last visited on 16 July 2009.
- *Fermi Questions—General Collection* in Sheila Talamo's site of the *Math Forum*. In http://mathforum.org/workshops/sum96/interdisc/fermicollect.html, last visited on 04 April 2012.

- *Problem B:* How many people are attending a demonstration?
- *Problem C:* How many drops of water do we need to fill a whole bucket?
- *Problem D:* How many glasses of water do we need to fill a swimming pool?
- *Problem E:* How many one-euro coins fit in a 1-m^3 safe-box?

All of these problems are related to the ideas of surface area and volume, which are familiar concepts for students of compulsory secondary education. Problems A and B require estimating the amount of people occupying a surface area, even though the surface in problem A is significantly smaller than that in problem B. In addition, in the first case, the surface has well-defined boundaries, whereas the surface area in the second case has yet to be determined. Problems C and D require estimating the number of objects that can be contained in a larger one. Students should be able to identify the relationship between the capacity of both items as the main issue to be addressed. Both problems come across as identical except for the size of the given objects; however, the pilot test revealed that a different contextualization allows for different ways of approaching them. Problem E introduces a similar situation, but as in the case of problems A and B, the object used to fill the volume does not completely cover all the available space. None of the five problems explicitly state the concepts of surface area or volume, but the answer for all of the cases lies in relating the question to its corresponding magnitude, in order to estimate the requested amounts reasonably.

When handing the problem to the student, it is important to stress that the student should only explain the way to approach the situation to obtain an approximate value. Each problem contextualizes its formulation within a situation that ascribes meaning to the question posed. The questions asked in the problems may not always seem realistic; for instance, problem D considers measuring a swimming pool using glasses of water. However, its formulation describes the context of a drought, and in this case, the context will be the main aspect to consider when studying this real-life problem.

Problem B, for instance, was introduced as follows:

> When there is a demonstration it is important to find out how many people have attended in order to evaluate the strength of the cause. Sometimes, the police, the local authorities and the organization of the demonstration cannot come to an agreement on the figures. If there were a demonstration going on, how would we be able to find out how many people had attended?
> Describe the steps you would follow to approximately calculate this quantity **with your own resources**. You are not required to provide a result, just to explain how you would reach it.

3.2 Data collection

Data gathering was carried out in two secondary schools in a city of 200,000 inhabitants within the metropolitan area of Barcelona. One of the educational institutions is a state school, and the other is a private one, both located in an area of medium socioeconomic level. The surveyed pupils were at the level of compulsory secondary education, which includes all students between 12 and 16 years of age. The students had no prior preparation specific to solving this kind of problem.

Data collection was carried out during the mathematics lessons, in 1-h sessions. Each student was handed a worksheet with a problem, and they had to describe the scheme to solve it. When students completed their scheme, the person in charge of collecting the data provided them with another worksheet. We therefore have 448 schemes from 216 students distributed by institution and level, as shown in Table 1.

Table 1 Number of participating students (ESO: compulsory secondary education)

Year (level of studies)	School 1	School 2	Total
1st ESO (12–13 years old)	30	17	47
2nd ESO (13–14 years old)	26	24	50
3rd ESO (14–15 years old)	30	25	55
4th ESO (15–16 years old)	26	38	64
Total	112	104	216

Table 2 shows the number of worksheets gathered by problem and year.

The way the data were gathered ensured a similar amount of worksheets for each problem. On the other hand, since the students of higher levels answered their worksheets in less time, the number of worksheets collected for each level increased with age. However, this does not interfere with the results of our analysis since the age variable was not considered in this study.

3.3 Data analysis

The schemes students devised to solve the problems constitute the data for this study. The aim of the analysis was to characterize the strategies described by the students and discuss to what extent they are suitable to solve the problem raised.

Schoenfeld (1985) argues that part of the difficulty of teaching strategies and heuristics to students is that a strategy may evolve in different ways for different problems. On a first reading of the data, we realized that sometimes the schemes devised to solve different problems correspond to the same approach but are specified in different ways according to the different situations. This leads us to distinguish between *strategies* and *actions*. By strategy, we understand "a plan of action or policy designed to achieve a major or overall aim"; a strategy represents a generic way of acting in order to reach a certain goal. A strategy is a general way to proceed, for instance, *counting all of the objects one by one* or *breaking the problem up into smaller subproblems*. If we focus on the procedures applied to solving a specific situation, we consider *actions* as the specification of a strategy for a concrete problem, for example, *writing down the objects on a list* or *checking them off while counting them*.

The first part of the analysis consists of identifying the action types suggested in each worksheet, to inductively create a group of categories that describe the procedures involved in the plans to solve the problems. In order to do this, we firstly identify the action proposed in each of the worksheets. Some of the actions identified are common to different problems (such as counting the amount of people in a given space). Other actions have been proven to be different forms of the same type of action plan, adapted to different contexts in which the problem is formulated. For instance, calculating the surface taken up by one person in the school playground and comparing this with the total surface area of the site to find out how many of them would fit

Table 2 Number of gathered worksheets

Year (level of studies)	PA	PB	PC	PD	PE	Total
1st ESO (12–13 years old)	18	18	19	18	18	111
2nd ESO (13–14 years old)	22	21	20	21	21	127
3rd ESO (14–15 years old)	25	22	24	22	20	133
4th ESO (15–16 years old)	31	31	25	25	27	167
Total	96	92	88	86	86	448

are different types of actions but use the same principles as finding the volume of a coin and comparing it to the volume of a safe-box. However, these actions can be grouped into the same category, since they answer to a certain type of strategy: to see how many times a smaller object fits into a larger one. The analysis process is developed recursively, and we consider the group of categories generated as complete when the proposal of each new worksheet analyzed is included in a pre-existing category, thus eliminating the need to create new categories. Therefore, each category groups different types of actions in the form of the same general strategy.

This analysis has been backed up with the use of the software Nvivo8 which, amongst other things, enables the codification of worksheets as a function of identified actions, as well as grouping the actions into different categories according to the type of strategy. This software is also useful in the validation process of the analysis, since it lists all the proposals codified in the same category and checks the consistency of the codification.

The following section, which features the categories created, provides examples and accounts for some of the actions identified, furnishing the reader with more information on the process of analysis.

4 Results

We firstly introduce the characterization of the different kinds of strategies detected in the students' plans to solve the problems, describing them and giving examples for them. Secondly, we discuss to what extent these plans would be effective to solve the problems set, thus responding to the two main aims of this study.

4.1 Characterization of strategies

Our process of analysis involves constructing categories in order to organize the schemes created by the students to solve the problems, taking note of the strategies they contain. The set of categories is our first result, since it comprises the characterization of the collected strategies. These categories are as follows:

- A well-defined strategy cannot be identified.
- Information from an external source is required.
- Exhaustive count
- Reduction of the problem and use of proportion
- Use of concentration measures
- Iteration of the measurement of a base unit of reference
- Use of a grid distribution model

For each of the categories, we present their characterization and give examples with elements from the students' submitted work, translated from Catalan and trying to preserve their expressions as much as possible. Each example states which problem it belongs to and the student's level.

4.1.1 A well-defined strategy cannot be identified

A large part of the students' proposals do not actually answer the question posed, since they are unfinished or are unintelligible. Others describe processes that are not related to the unknown quantity. Hence, this category contains the students' schemes to solve the problems which attempt to account for a procedure designed to estimate the required quantity but fail to describe the process specifically enough, preventing us from detecting a clear strategy.

An example for problem C of a fourth grade student would be (problem C/drops of water/fourth year) “It depends on the size of the drops of water and on the rate at which the drops fall.” Here, we observe that the student does not present a specific scheme to solve the problem but only mentions some of the elements he considers important.

4.1.2 Information from an external source is required

This category groups students' schemes aimed at solving the problem by transferring the question to an external source of information. The schemes they describe are centered on specifying which source to use and what information is required.

An example would be (problem B/demonstration/third year) “You call the police and ask them.” The only plan found in this submission is asking an external source for information, which would usually provide this kind of details to the media.

4.1.3 Exhaustive count

This category includes the schemes of the students whose main strategy is to manually or automatically count the required objects. This category is the first one that genuinely includes schemes to solve the problems. Therefore, it allows us to introduce the distinction between actions and strategies in the analysis, as mentioned in the previous section.

For instance, the following scheme is based on counting people using a mechanical counting artifact (problem B/demonstration/third year): “I would place a sensor shaped like a plate in the area where the demonstration starts, so that when people tread on it, it counts them.” We observe that the students' knowledge of different counting technologies (like turnstiles in closed premises) led them to consider this a valid option for counting the people attending a demonstration. Similarly, we also found students who suggested to use a video camera which captures drops of water falling into the bucket for problem C or to take an aerial photograph of the high school playground full of people.

Another example would be (problem D/glasses of water in a swimming pool/third year):

> I would take some glasses and start filling them up with water from the swimming pool. If I needed more glasses I would take them and keep going until there is no water left in the pool. When the water is finished I would count the glasses filled and then I would know how many glasses I need.

In this case, the scheme is based on directly counting the glasses you need to empty a swimming pool without considering the effort involved and the time and resources it requires.

In this category, we have also included the schemes of students who expect to mark the objects, count them, or create a list with the names of people who are attending a demonstration. Counting people with a device, drops of water with a video recorder, people with an aerial photograph, counting the number of glasses of water, or making a list of people are different actions based on a single strategy, different versions of exhaustive counting being inspired by the context formulated in the problem.

4.1.4 Reducing the problem and using proportion

This category includes the plans submitted by students who reduce the problem to a smaller version and suggest applying a scale factor to it in order to obtain an estimate of the required amount. An example of this would be (problem C/drops of water in a bucket/third year):

> I would calculate the amount of water that fits in a bucket. After that, I would divide it by a hundred and calculate the number of drops needed to fill a hundredth of the bucket. Then I would multiply this number by a hundred.

The difficulty of the problem lies in the fact that a bucket can hold a huge number of drops. Hence, the student has sought an equivalent problem which is easier to solve by simplifying this difficulty, while keeping track of the scale factor used.

An equivalent way to approach problem A is the following (problem A/concert/third year):

> Everything depends on whether the people are very close together or not. I would calculate the amount of people that fits in an area of the playground. I would then multiply the result of the number of people in that small area to reach the total surface area.

In this scheme, the student aims to limit an area where it is feasible to directly count the number of people contained therein, reducing the problem to a smaller one. From his words, we can infer that the student intends to multiply the result by the proportion factor that relates the total surface of the playground to the total surface of the chosen area. The last part has not been stated rigorously or clearly, but in the analysis, we interpret that was his intention.

4.1.5 Using concentration measurements

By concentration measurement, we understand the ratio between the number of objects distributed in a given space and the amount of space they take up. An example detected in the students' schemes corresponds to the idea of population density (number of people per square meter). Hence, one of the schemes presented for problems A (concert) and B (demonstration) is to determine the total surface area of the place where the people are congregated and multiply it by the population density calculated experimentally.

The following is an example of this procedure (problem A/concert/first year):

> I would take a group of people and find out how many of them fit into a square metre of the playground. I would then measure the surface of the playground in square metres and multiply that by the number of people that fit in a square metre. That would be the solution.

We have also detected the use of concentration measurements for other problems (problem D/glasses of water in a swimming pool/second year):

> Well it depends on the volume of the swimming pool and of the capacity of the glass. If you have time to kill, you can fill up the swimming pool glass by glass. If you would like to go faster, you could calculate the volume of the swimming pool in cubic metres and how many glasses you would need to fill up a cubic metre. You then multiply the two results and you obtain the number of glasses of water you need.

In this case, the concentration measurement suggested is the number of glasses per cubic meter. The other type of concentration measurement identified is the number of drops that have fallen in a certain amount of time, in the schemes for problem C (drops of water in a bucket).

4.1.6 Iteration of the measurement of the base unit of reference

This category results from grouping proposals which display schemes that are based on determining the total measurement of the place where the objects to be counted are placed

and divide it by the measurement of one only object; that is, they estimate the number of elements by introducing a reference unit. Some of the reference units used are the surface area a person takes up, the volume of a drop of water or the volume of a coin.

The suggested unit is an abstraction of the concept of the element that has to be counted, since, except in the case of the coins in the safe, we cannot consider that the different elements to be counted are exactly the same size. The use of this reference unit is directly related to the *iteration of a unit*, which is a strategy of length estimation which consists of applying the mental image of a length unit several times (centimeter, meter…) to an object in order to estimate its length.

An improvement to the iteration of a unit is to introduce an everyday-life object with a well-known length in order to estimate the length of a different object. This strategy is known as iteration of a reference point, where the reference point is an object the estimator considers a good unit of measurement (Carter, 1986). An example of the use of the base reference unit would be (problem D/glasses of water in a swimming pool/second year): "It can't be done because we don't have enough information. If we did, we would be able to divide the capacity of the swimming pool by the capacity of a glass and that would give us the number of glasses." In this case, the student cannot directly apply the image of the glass over that of the swimming pool, but this image allows her to establish how the number of glasses could be approximated by division. Even though the student's answer is rather short and does not specify the particular steps to be followed, this student gives enough information to consider that her scheme is based on finding the volume of the swimming pool and the capacity of a glass.

An equivalent action plan would be (problem A/concert/first year): "I would calculate how big the playground is and look for a person who is neither fat nor thin. Then we can make a division to obtain how many people would fit." We consider these two schemes equivalent, because if we substitute the word swimming pool with playground and glass with person, they involve applying the same strategy. Therefore, this is an example of different actions applied to different contexts which represent the same type of strategy.

4.1.7 Use of a grid distribution model

This category includes the students' schemes which suggest solving a given situation by means of a grid distribution of the objects to be quantified, estimating the number of objects that each dimension can contain in order to multiply the resulting values. An example of this would be (problem A/concert/first year):

> First I would mark the area of the stage, then I would place a row of chairs along the playground to its end and I would count them—After that I would do the same but along the width of the playground. I would finally multiply the length by the width and the resulting number will be the number of tickets.

In this case, the student imagines an arrangement of chairs on a flat grid, since the problem refers to a surface distribution. The scheme involves breaking the problem into two smaller problems that can be more readily solved.

Another scheme included in this category would be (problem E/safe/second year):

> I would make a pile of one-euro coins until it reaches the top, the top edge of the safe-box (and I would count them). Then I would arrange the coins horizontally on the bottom of the box until I can't get any more in and I would count them. I would multiply both amounts and that would be the solution.

This action plan focuses on finding out the number of coins that comprise a pile in the safe-box, as a first sub-problem, in order to multiply them by the result of covering the bottom of the box with coins, as a second sub-problem. For this problem, we have also detected schemes that aim to determine the number of coins that can be placed on each of the three dimensions of the box and multiply the three numbers, which are directly related to the concept of volume.

4.2 Adapting the strategies to the problem

In this section, we present the results related to our second goal, discussing to what extent the schemes presented by the students would be suitable to solve the problems. Table 3 shows how the different types of strategies are distributed in the gathered worksheets.

As can be observed in Table 3, the schemes with poorer content—without a strategy, external source, or exhaustive count—represent 52.9 % of those collected. The strategies that require an external source or those that suggest an exhaustive count do not allow for working on mathematical content in class. However, they can be used to trigger a discussion on what solving a mathematical problem means.

The remaining 47.1 % corresponds to more mathematically interesting strategies: the reduction to a smaller problem and then applying a proportion factor, the use of concentration measurements, the iteration of a base unit as a reference, and the grid distribution model. All of these have been detected in several of the schemes collected, a fact which shows that they are flexible strategies, adaptable to different situations, by customizing the actions to the scheme.

Table 4 gathers the strategies collected for each problem.

The table displays some values which stand out—for example, the grid distribution can only be detected in problems A, B, and E, which aim to easily count separable objects. On the other hand, this cannot be found in problems C and D, where the context of the problem refers to water, a continuous material, which is difficult to regard as drops or glasses. Hence, we observe that the strategy is adapted to the context set out in the problem.

We can also see that problems with an equivalent mathematical approach but a different context give rise to different strategies. This is the case of problems A and B for the strategy of the reference point. While in the case of the high school playground, almost a fourth (24.0 %) of the action plans include this strategy, and in the case of the demonstration, it is only present in 4.3 % of the schemes. This type of strategy has also been used more in problem D, with a higher percentage than any other strategy. This is possibly due to the fact that the concept of "capacity of a glass" is more within the students' reach than "the volume in a drop of water," its equivalent in problem C. For problem C, the students prefer to reduce the problem to a smaller version of it. This stresses the idea that the knowledge of a real situation inspires the students to

Table 3 Number of worksheets by strategy and percentage against the total

Strategy	Schemes	Percentage (%)
Without strategy	139	31.0
External source	9	2.0
Exhaustive count	89	19.9
Reduction and proportion	30	6.7
Concentration measurement	38	8.5
Reference point	128	28.6
Grid distribution	15	3.3
Total	448	100.0

Table 4 Number and percentage of the incidence of each type of strategy for each problem

Strategy	PA	PB	PC	PD	PE
Without strategy	37 (38.5 %)	28 (30.4 %)	25 (28.4 %)	17 (19.8 %)	32 (37.2 %)
External source	3 (3.1 %)	4 (4.3 %)	0 (0.0 %)	2 (2.3 %)	0 (0.0 %)
Exhaustive count	16 (16.7 %)	35 (38 %)	9 (10.2 %)	10 (11.6 %)	19 (22.1 %)
Reduction and proportion	3 (3.1 %)	2 (2.2 %)	21 (23.9 %)	2 (2.3 %)	2 (2.3 %)
Concentration measurement	12 (12.5 %)	14 (15.2 %)	9 (10.%)	3 (3.5 %)	0 (0.0 %)
Reference point	23 (24.0 %)	4 (4.3 %)	24 (27.3 %)	52 (60.5 %)	25 (29.1 %)
Grid distribution	2 (2.1 %)	5 (5.4 %)	0 (0.0 %)	0 (0.0 %)	8 (9.3 %)
Total	96	92	88	86	86

elaborate their action plans since the context provided in the formulation would be an important element to help the students create schemes.

To conclude this section, we aim to find out to what extent the different strategies found in the schemes could be useful to solve the problems presented. In order to determine the success of each of these plans, we have considered each one to be carried out by a competent problem solver, and we have evaluated the type of solution the solver would reach. Three options have been distinguished in the analysis: either the plan would enable solving the problem, or it would not lead to solve the problem, or it would solve the problem "on paper only," the proposed plan being unfeasible.

We therefore consider that a scheme would solve the problem if it is specific enough for a competent solver to follow it and reach a result following mathematically correct and valid processes. We consider that a plan would not solve a problem if it is incomplete, uses concepts in the wrong way, has relevant information gaps in its description, or contains calculation errors. Finally, we decide to incorporate a third category, the unfeasible proposals, which includes those schemes which solve the problem "on paper only," those plans that are not conceptually incorrect but cannot be put into practice due to a lack of resources or time, as in the cases in which the student suggests to count all the elements involved, which would require too much time. In this sense, we render these proposals unfeasible for a competent problem solver.

Table 5 displays the results on the success of the scheme for each of the identified strategies.

The results obtained show the successful strategies secondary school students can come up with without any previous specific training. These strategies are rich in conceptual content. Thus, Table 5 shows that the strategies which are linked to successful schemes are the

Table 5 Success in the solution plan for each of the strategies detected

Strategy	Solves	Unfeasible	Does not solve	Total
Without strategy	0	0	139	139
External source	0	4	5	9
Exhaustive count	0	80	9	89
Reduction and proportion	12	4	14	30
Concentration measurement	17	11	10	38
Reference point	63	18	47	128
Grid distribution	11	0	4	15
Total	103	117	228	448

reduction of a problem to a smaller one, the use of concentration measurements, and the use of the reference point and the grid distribution. Other valid strategies may exist for solving these problems; however, we have found no evidence of this in the students' worksheets and therefore cannot expect them spontaneiously to come up with such strategies.

Our analysis also shows that there are students who base their plan on strategies that could be successful but cannot get to adapt them completely to the problem's needs. By examining their proposals, we observed that some of them embark on an unfinished path or show a lack of mathematical understanding by, for instance, making mistakes in volume and surface calculations. We also observe that in problems involving estimating large quantities, exhaustive counts only help to solve the problem "on paper" and the aid of an external source does not improve the practical results. Particularly, we observe that the students consider the possibility of using external information sources but do not know how to obtain valid information in the required contexts.

5 Concluding remarks

We have shown that almost half of the students in our study, of different ages, without any experience in this type of problems, are able to create schemes which include mathematical strategies that are suitable for the problems presented. We would like to point out that focussing on the students' action plans has enabled us to study the resolution strategies in more detail. However, we believe that the rate of successful resolutions would have increased if the students had been allowed to implement their plans. Therefore, what seems like a limitation from a teaching point of view has proved useful in research, allowing us to identify features that would otherwise be unclear.

We have seen that the students can elaborate high-level strategies and they would undoubtedly be able to come up with other kinds of equally interesting strategies by working on a wider range of problems. Based on the data obtained, we conclude that Fermi problems oriented to the estimation of large quantities offer possibilities for working in class, since they enable the introduction of problem solving strategies which include the use of relevant mathematical concepts. If these problems are addressed in teams, the students can share their proposals and elaborate their own strategies and adapt them to complex realities, as we show in Albarracín and Gorgorió (2013).

We observe that the nature of the problems suggests a great variety of strategies to the students which are specified in different kinds of actions adapted to the situations and contexts introduced in each problem. The mathematical strategies detected prompt schemes comprising different actions which respond to equivalent global plans. Therefore, we claim that Fermi problems aimed at estimating large quantities answer to Schoenfeld's (1992) call for the development of heuristics and specific solution strategies that are clearly related to different types of problems. The richness of the strategies collected and of the mathematical concepts which come up in the students' answers provides ideas to the teachers about what they can expect from their students when solving similar problems. In addition, the distinction between actions and strategies might facilitate the teachers' task of finding situations that promote the use of a certain strategy.

Another feature which makes the problems of this study interesting arises from the fact that they are set in situations involving large numbers. Working on the reduction to a smaller problem is a good chance to consolidate the concept of proportion factor. Some of the strategies identified are related to estimation strategies used for numerosity or magnitude estimations, but other strategies detected correspond to modeling processes of a real situation

that can be related to other school subjects. Thus, the iteration of a unit as a reference point is a type of strategy that is used for estimating lengths, which in this study appears in situations where a whole is compared to a part. The use of concentration measurements can be related to the content of other subjects the students may be studying (population density in Geography or the concentration of solutions in Chemistry). Moreover, the grid distribution model is related to the rule of the product which is introduced to justify the surface area of a rectangle and can be extrapolated to define the volume of a cube.

We cannot either ignore the relevance of the proposals which present plans which solve the problems "only on paper." They offer us a new avenue for researchers to explore: to study how teachers find ways to help students to move from this kind of proposals to more conceptually adequate ones, while taking advantage of students' potentialities. In this sense, it is possible that working in small teams and using different technologies may lead students to approach the context of the problems from a different point of view. These tools may help them to employ the knowledge demonstrated when devising unfeasible plans and relate it to desirable strategies that would lead to more suitable concepts and higher levels of success.

In order to analyze the strategies presented by the students, it has proved invaluable to exclusively study the output relative to the stage of devising a plan. By focusing on this stage, we have obtained explicit data which are not easy to obtain from a complete description of the solving process, as already documented by Van Dooren, De Bock, Vleugels, and Verschaffel (2010). From this point of view, we follow the trend of Leong, Toh, Tay, Quek, and Dindyal (2012), who suggest going into each of the stages proposed by Pólya for problem solving, emphasizing the planning of the task, which can enable the students to monitor their own performance more effectively.

Our study allows us to consider that Fermi problems oriented to estimating large amounts can be approached in a way that connects school mathematics with aspects related to the students' environment. Hence, problems based on social issues and everyday-life situations can be worked on, which are an essential element of an educational agenda which aims to educate mathematically competent citizens, since the class activity directly approaches heuristics that the students are expected to apply as a citizen in the future. The problems, their formulations, and the solving strategies should be close to the objective they aim to reach. Nonetheless, working on Fermi problems oriented towards estimating large quantities may aid students to create ways of explaining their surrounding phenomena on the basis of their own mathematical knowledge adapting it to familiar situations, establishing the foundations of critical thinking.

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References

Albarracín, L., & Gorgorió, N. (2012). On strategies for solving inconceivable magnitude estimation problems. In T. Y. Tso (Ed.), *Proceedings of the 36th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 11–18). Taipei: PME.

Albarracín, L., & Gorgorió, N. (2013). Fermi problems involving big numbers: Adapting a model to different situations. *To appear in proceedings of the 8th congress of the European Society for Research in Mathematics Education (CERME8)*. Antalya: CERME.

Arcavi, A., & Friedlander, A. (2007). Curriculum developers and problem solving: The case of Israeli elementary school projects. *ZDM The International Journal on Mathematics Education, 39*, 355–364.

Ärlebäck, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Montana Mathematics Enthusiast, 6*(3), 331–364.

Barth, H., Kanwisher, N., & Spelke, E. (2003). The construction of large number representations in adults. *Cognition, 86*, 201–221.

Booth, J. L., & Siegler, R. S. (2006). Developmental and individual differences in pure numerical estimation. *Developmental Psychology, 41*(6), 189–201.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45–75.

Carter, H. L. (1986). Linking estimation to psychological variables in the early years. In H. L. Schoen & M. J. Zweng (Eds.), *Estimation and mental computation* (pp. 74–81). Reston, VA: NCTM.

Chapman, O. (2006). Classroom practices for context of mathematics word problems. *Educational Studies in Mathematics, 62*, 211–230.

Efthimiou, C. J., & Llewellyn, R. A. (2007). Cinema, Fermi problems and general education. *Physics Education, 42*(42), 253–261.

Ginsburg, N. (1991). Numerosity estimation as a function of stimulus organization. *Perception, 20*(5), 681–686.

Hanson, S. A., & Hogan, T. P. (2000). Computational estimation skill of college students. *Journal for Research in Mathematics Education, 31*, 483–499.

Hildreth, D. J. (1983). The use of strategies in estimating measurements. *Arithmetic Teacher, 30*(5), 50–54.

Hogan, T. P., & Brezinski, K. L. (2003). Quantitative estimation: One, two, or three abilities? *Mathematical Thinking and Learning, 5*(4), 259–280.

Jones, M. G., Gardner, G. E., Taylor, A. R., Forrester, J. H., & Andre, T. (2012). Students' accuracy of measurement estimation: Context, units, and logical thinking. *School Science and Mathematics, 112*(3), 171–178.

Jones, M. G., Taylor, A., & Broadwell, B. (2009). Estimating linear size and scale: Body rulers. *International Journal of Science Education, 31*(11), 1495–1509.

Lemaire, P., & Lecacheur, M. (2002). Children's strategies in computational estimation. *Journal of Experimental Child Psychology, 82*, 281–304.

Leong, Y. H., Toh, T. L., Tay, E. G., Quek, K. S., & Dindyal, J. (2012). Relooking "look back": A student's attempt at problem solving using Pólya's model. *International Journal of Mathematical Education in Science and Technology, 43*(3), 357–369.

Lesh, R., & Harel, G. (2003). Problem solving, modeling, and local conceptual development. *Mathematical Thinking and Learning, 5*(2), 157–189.

Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763–804). Charlotte: Information Age Publishing.

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education, 25*(6), 660–675.

Peter-Koop, A. (2004). Fermi problems in primary mathematics classrooms: Pupils' interactive modelling processes. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010 (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Townsville* (pp. 454–461). Sydney: MERGA.

Pólya, G. (1945). *How to solve it*. Princeton: Princeton University Press.

Reys, R. E., Rybolt, J. F., Bestgen, B. J., & Wyatt, J. W. (1982). Processes used by good computational estimators. *Journal for Research in Mathematics Education, 13*, 183–201.

Ross, J., & Ross, M. (1986). Fermi problems or how to make the most of what you already know. In H. L. Schoen & M. J. Zweng (Eds.), *Estimation and mental computation* (pp. 175–181). Reston, VA: NCTM.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. San Diego, CA: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970-2008: Research and theory, practice and politics. *ZDM The International Journal on Mathematics Education, 39*(5–6), 537–551.

Sophian, C., & Chu, Y. (2008). How do people apprehend large numerosities? *Cognition, 107*, 460–478.

Van den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics, 25*(2), 2–10.

Van Dooren, W., De Bock, D., Vleugels, K., & Verschaffel, L. (2010). Just answering…or thinking? Contrasting pupils' solutions and classifications of proportional and non-proportional missing-value word problems. *Mathematical Thinking and Learning, 12*(1), 20–35.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Lisse, The Netherlands: Swets and Zeitlinger Publishers.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–628). Charlotte, VA: Information Age Publishing.